Categorifying the Knizhnik-Zamolodchikov connection

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Categorification of link invariants

Context-Categorification of link invariants

- Quantum link invariants can be defined:
  - Combinatorially (via quantum group R-matrices)
  - Analytically (via the holonomy of the Knizhnik-Zamolodchikov connection)

- Most approaches to categorification of quantum link invariants use combinatorial frameworks.

- It seems however natural to use differential-geometric approaches for categorifying quantum link invariants.

- Main aim of this project: Define a 2-connection categorifying the Knizhnik-Zamolodchikov connection.
The Knizhnik-Zamolodchikov connection

- The configuration space $\mathbb{C}(n)$ of $n$ (distinguishable) particles in the complex plane is

$$\mathbb{C}(n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}.$$

- Let $\mathfrak{g}$ be a Lie algebra.
- Let $\langle, \rangle$ be a $\mathfrak{g}$-invariant, non-degenerate, symmetric, bilinear form in $\mathfrak{g}$.
- Let $\{s_i\}$ be basis of $\mathfrak{g}$.
- Let $\{t^i\}$ be the dual basis of $\mathfrak{g}^* \cong \mathfrak{g}$.
- Let

$$r = \sum_{i} s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$$

- Note that $r$ is symmetric: $r_{12} = r_{21}$.
- Choose a representation of $\mathfrak{g}$ on a vector space $V$. 

- Background
  - The Knizhnik-Zamolodchikov connection
  - Crossed modules and differential crossed modules
  - Complexes of vector spaces and differential crossed modules
  - Categorifying the Knizhnik-Zamolodchikov connection
  - Infinitesimal 2-R-matrices
The Knizhnik-Zamolodchikov connection is given by the Hom(V^n⊗)-valued 1-form A in the configuration space $\mathbb{C}(n)$,

$$A = \frac{h}{2\pi i} \sum_{1 \leq a < b \leq n} \omega_{ab} \phi_{ab}(r),$$

Where

$$\omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}$$

and $\phi_{ab}(r): V^\otimes n \rightarrow V^\otimes n$ is the linear map such that:

$$\phi_{ab}(r)(v_1 \otimes \ldots \otimes v_a \otimes \ldots \otimes v_b \otimes \ldots \otimes v_n) = \sum_i v_1 \otimes \ldots \otimes s_i \triangleright v_a \otimes \ldots \otimes t_i \triangleright v_b \otimes \ldots \otimes v_n,$$

where $r = \sum_i s_i \otimes t_i \in g \otimes g$. 
The 4-term relation

- The Knizhnik-Zamolodchikov connection is flat, that is the curvature 2-form $F_A = dA + \frac{1}{2} A \wedge A$ vanishes.
- This follows from the $\mathfrak{g}$-invariance of $\langle \cdot, \cdot \rangle$, which implies the relation (known as the 4-term relation):
  \[
  [r_{12} + r_{13}, r_{23}] = 0, \quad \text{in } \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}),
  \]
  where
  \[
  r_{12} = \sum_i s_i \otimes t_i \otimes 1, \quad r_{13} = \sum_i s_i \otimes 1 \otimes t_i, \quad r_{23} = \sum_i 1 \otimes s_i \otimes t_i.
  \]
- Explicitly:
  \[
  \sum_{i,j} s_i \otimes [t_i, s_j] \otimes t_j + \sum_{i,j} s_i \otimes t_j \otimes [s_i, t_j] = 0.
  \]
- Such a symmetric tensor $r = \sum_i s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$ will be called an infinitesimal R-matrix in $\mathfrak{g}$. (Possibly a better designation would be infinitesimal Yang-Baxter operator).
The 4-term relation and the symmetry of $r$ imply that:

$$
\phi_{ab}(r)\phi_{bc}(r) + \phi_{ac}(r)\phi_{bc}(r) = \phi_{bc}(r)\phi_{ab}(r) + \phi_{bc}(r)\phi_{ac}(r),
$$

$$
\phi_{ab}(r) = \phi_{ba}(r),
$$

for each distinct $a, b, c \in \{1, \ldots, n\}$.

We also have:

$$
[\phi_{ab}(r), \phi_{a'b'}(r)] = 0, \text{ if } \{a, b\} \cap \{a', b'\} = \emptyset.
$$

These relations will be called \textit{infinitesimal braid group relations}.

Compare with the usual braid group relations:

$$
X_aX_{a+1}X_a = X_{a+1}X_aX_{a+1}
$$

$$
X_aX_b = X_bX_a, \text{ if } |a - b| \geq 2.
$$
Drinfeld-Kohno Theorem

- Recall that the braid group $B_n$ is isomorphic to the fundamental group of $\mathbb{C}(n)/S_n$, where the symmetric group $S_n$ acts on
  \[ \mathbb{C}(n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\} \]
  by permutation of coordinates.
- The Knizhnik-Zamolodchikov connection $A$ is invariant under the action of the symmetric group.
- Therefore we have a quotient Knizhnik-Zamolodchikov connection $A$ in the quotient vector bundle
  \[ (\mathbb{C}(n) \times V^{n\otimes})/S_n. \]
- Given that $A$ is flat, by considering its holonomy, we have a group morphism:
  \[ \text{Hol}_{\text{KZ}} : \pi_1(\mathbb{C}(n)/S_n) \cong B_n \to \text{GL}(V^{n\otimes}). \]
Drinfeld-Kohno Theorem

Theorem (Drinfeld-Kohno)

If \( \mathfrak{g} \) is semisimple, \( \langle , \rangle \) is the Cartan-Killing form, and \( V \) is a representation of \( \mathfrak{g} \), then the representation of the braid group \( B_n \) given by the holonomy of the Knizhnik-Zamolodchikov connection is equivalent to the representation of the braid group \( B_n \) coming from the \( R \)-matrix in \( U_h(\mathfrak{g}) \) and the action of \( U_h(\mathfrak{g}) \) on \( V_h \) (the quantisation of \( V \)).

- The holonomy of the Knizhnik-Zamolodchikov connection cannot be immediately extended to links in \( S^3 \), since the forms \( \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b} \) explode at maximal and minimal points.
- There exist regularisation techniques for the holonomy at extreme points, and this leads to the usual quantum group link invariants (and the Kontsevich Integral).
Consider the Lie algebra \( \mathfrak{ch}_n \), generated by the symbols \( r_{ab} \), where \( 1 \leq a, b \leq n \), satisfying the infinitesimal braid group relations

\[
\begin{align*}
    r_{ab} &= r_{ba}, \\
    [r_{ab}, r_{cd}] &= 0 \text{ for } \{a, b\} \cap \{c, d\} = \emptyset, \\
    [r_{ab} + r_{ac}, r_{bc}] &= R_{abc} = 0.
\end{align*}
\]

Call it the *Lie algebra of horizontal chord diagrams in \( n \)-strands*.

Consider the connection form in \( \mathbb{C}(n) \)

\[
A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab}.
\]

As before

\[
\omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}.
\]
The Kontsevich integral

- The holonomy of $A$ takes values in the space of formal power series over the universal enveloping algebra $\mathcal{U}(\text{ch}_n)$ of $\text{ch}_n$.
- This holonomy can be regularised at maximal and minimal points of embedded links, defining a link invariant with values in the space of formal power series in the Hopf algebra of chord diagrams in the circle.
- This is called the (framed) Kontsevich integral, and can be proven to be a universal Vassiliev invariants of links.
Main aim of this work:

- **Categorify** the Knizhnik-Zamolodchikov connection in order to (possibly) obtain invariants of braid cobordisms.
- **Categorify** the Lie algebra of horizontal chord diagrams.
- Discuss the *infinitesimal relations* for braid cobordisms.
- **Categorify** the notion of an infinitesimal R-matrix in a Lie algebra.
- Find examples.
- Setting: 2-connections on 2-bundles.
- Particular case considered here: 2-connections on vector bundles with typical fibre being a chain complex of vector spaces.
Definition (Lie crossed module)

A Lie crossed module:

\[ \mathcal{G} = (\partial: H \to G, \triangleright) \]

is given by a Lie group morphism \( \partial: H \to G \) together with a smooth left action \( \triangleright \) of \( G \) on \( H \) by automorphisms, such that the following relations, called Peiffer relations, hold:

1. \( \partial(g \triangleright h) = g\partial(h)g^{-1}; \) for each \( g \in G \) and \( h \in H \),
2. \( \partial(h) \triangleright h' = hh'h^{-1}; \) for each \( h, h' \in H \).

The category of crossed modules is equivalent to the category of strict 2-groups (Brown and Spencer).
Differential crossed modules

The differential counterpart of a Lie crossed module is what is called a differential crossed module.

**Definition (Differential crossed module)**

A differential crossed module:

\[ \mathfrak{G} = (\partial: \mathfrak{h} \to \mathfrak{g}, \triangleright) \]

is given by a Lie algebra morphism \( \partial: \mathfrak{h} \to \mathfrak{g} \) together with a left action of \( \mathfrak{g} \) on \( \mathfrak{h} \) by derivations, such that the following relations, also called Peiffer relations, hold:

1. \( \partial(X \triangleright \xi) = [X, \partial(\xi)]; \) for each \( X \in \mathfrak{g} \), and each \( \xi \in \mathfrak{h} \),
2. \( \partial(\xi) \triangleright \nu = [\xi, \nu]; \) for each \( \xi, \nu \in \mathfrak{h} \).

The category of differential crossed modules is equivalent to the category of strict Lie-2-algebras (Baez and Crans).
We can construct a differential crossed module

\[ \mathfrak{gl}(V) = (\beta : \mathfrak{gl}^1(V) \to \mathfrak{gl}^0(V), \triangleright) \]

from any complex of vector spaces

\[ V = (\ldots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \ldots). \]

- Define a Lie algebra \( \mathfrak{gl}^0(V) \), given by all chain maps \( f : V \to V \), with the usual commutator of chain-maps.
- There exist two natural Lie algebra structures on the vector space \( \text{Hom}^1(V) \) of degree 1 maps \( V \to V \): where:

\[
\{s, t\}_l = s\partial t - t\partial s + st\partial - ts\partial,
\]

\[
\{s, t\}_r = s\partial t - t\partial s + \partial st - \partial ts.
\]
- There exists a Lie algebra map \( \beta : \text{Hom}^1(V) \to \mathfrak{gl}^0(V) \) with

\[ \beta(s) = \partial s + s\partial. \]
There exists an action by derivations of $\mathfrak{gl}^0(\mathcal{V})$ on $\text{Hom}^1(\mathcal{V})$ such that: $f \circ s = fs - sf$.

We do not always have differential crossed modules since the relation $\{s, t\} = \beta(s) \triangleright t$ may fail in general, unless we are considering a chain complex of length two.

Consider the map $\beta' : \text{Hom}^2(\mathcal{V}) \to \text{Hom}^1(\mathcal{V})$ such that

$$\beta'(h) = -h\partial + \partial h.$$ 

Then $\beta'(\text{Hom}^2(\mathcal{V}))$ is a $\mathfrak{gl}^0(\mathcal{V})$-invariant Lie algebra ideal of $\text{Hom}^1(\mathcal{V})$, for $\{,\}_l$ and $\{,\}_r$, contained in $\ker(\beta)$.

We have a quotient Lie algebra $\{,\}_l/ = \{,\}_r/$, in

$$\mathfrak{gl}^1(\mathcal{V}) = \frac{\text{Hom}^1(\mathcal{V})}{\beta'(\text{Hom}^2(\mathcal{V}))},$$ 

provided with a (quotient) map $\beta : \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V})$. 
Differential crossed modules from complexes of vector spaces

**Theorem:** Given a complex \( \mathcal{V} \) of vector spaces there exists a differential crossed module

\[
\mathfrak{gl}(\mathcal{V}) = (\beta : \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright).
\]

- Where \( \mathfrak{gl}^0(\mathcal{V}) \) is the Lie algebra of chain-maps \( \mathcal{V} \to \mathcal{V} \), with commutator

\[
[f, g] = fg - gf,
\]

- \( \mathfrak{gl}^1(\mathcal{V}) = \text{Hom}^1(\mathcal{V})/\beta'(\text{Hom}^2(\mathcal{V})) \) with commutator:

\[
\{s, t\} = s\partial t + st\partial - t\partial s - ts\partial,
\]

- \( \beta(s) = s\partial + \partial s \),

- \( \beta'(h) = -\partial h + h\partial \),

- \( f \triangleright s = fs - sf \).
A representation of a differential crossed module \( \mathcal{G} = (h \to g, \triangleright) \) on a complex of vector spaces \( \mathcal{V} \) is a differential crossed module map \( \rho: \mathcal{G} \to \mathfrak{gl}(\mathcal{V}) \)

\[
\mathcal{G} = (h \to g, \triangleright) \xrightarrow{\rho^1, \rho^0} (\beta: \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright) = \mathfrak{gl}(\mathcal{V}).
\]

For any \( X \in g \) we have a chain map \( \rho^0_X: \mathcal{V} \to \mathcal{V} \) and for each \( v \in h \) we have a chain homotopy (up to 2-fold homotopy) \( \rho^1_v \in \mathfrak{gl}^1(\mathcal{V}) \), such that:

1. \([\rho^0_X, \rho^0_Y] = \rho^0_{[X,Y]} \) where \( X, Y \in g \).
2. \( \{\rho^1_v, \rho^1_w\} = \rho^1_{[v,w]} \), where \( v, w \in h \).
3. \( \beta(\rho^1_v) = \rho^0_{\partial(v)} \), where \( v \in h \).

If there are representations \( \rho \) and \( \rho' \) of \( \mathcal{G} \) on \( \mathcal{V} \) and \( \mathcal{V}' \) then we have a representation of \( \mathcal{G} \) on \( \mathcal{V} \otimes \mathcal{V}' \):

1. \((\rho \otimes \rho')^0_X = \rho^1_X \otimes \text{id} + \text{id} \otimes \rho'^0_X \).
2. \((\rho \otimes \rho')^1_v = \rho^1_v \otimes \text{id} + \text{id} \otimes \rho'^1_v \).
Example Let \((\partial: \mathfrak{h} \rightarrow \mathfrak{g}, \triangleright)\) be a differential crossed module. The adjoint representation of \(\mathfrak{G}\) on its underlying chain complex \(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\) is given by the pair \(\rho = (\rho_1, \rho_0)\), where:

- If \(X \in \mathfrak{g}\) the chain map \(\rho_0^X: \mathfrak{G} \rightarrow \mathfrak{G}\) is such that
  \[
  \rho_0^X(Y) = [X, Y]
  \]
  and
  \[
  \rho_0^X(\zeta) = X \triangleright \zeta
  \]
  where \(Y \in \mathfrak{g}\) and \(\zeta \in \mathfrak{h}\).

- If \(\zeta \in \mathfrak{h}\) the homotopy \(\rho_1^\zeta: \mathfrak{g} \rightarrow \mathfrak{h}\) is such that
  \[
  \rho_1^\zeta(X) = -X \triangleright \zeta.
  \]
Let
\[ \mathcal{V} = (\ldots \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \ldots). \]
be a chain complex of vector spaces, with associated differential crossed module
\[ \mathfrak{gl}(\mathcal{V}) = (\beta : \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright) . \]

A local 2-connection \((A, B)\) in a manifold \(M\) is given by
- A 1-form \(A\) with values in \(\mathfrak{gl}^0(\mathcal{V})\).
- A 2-form \(B\) with values in \(\mathfrak{gl}^1(\mathcal{V})\).
- Such that \(\beta(B) = F_A = dA + \frac{1}{2} A \wedge A\).
- The 2-curvature of \((A, B)\) is, by definition:
  \[ \mathcal{M}_{(A,B)} = dB + A \wedge B. \]

Local 2-connections can be integrated to give a 2-dimensional holonomy.
The 2-category $\text{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

Let

$$\mathcal{V} = (\ldots \partial \rightarrow V_n \partial \rightarrow V_{n-1} \partial \rightarrow \ldots).$$

be a chain complex of vector spaces.

- Define a 2-category $\text{Aut}(\mathcal{V})$ with a single object.
- 1-morphisms: chain maps $f : \mathcal{V} \rightarrow \mathcal{V}$.
- Composition is done in the reverse order:

$$\mathcal{V} \xrightarrow{f} \mathcal{V} \xrightarrow{g} \mathcal{V} = \mathcal{V} \xrightarrow{fg} \mathcal{V}.$$

- The 2-morphisms $f \Rightarrow g$, have the form:

$$\begin{array}{c}
\mathcal{V} \\
\xrightarrow{f} \mathcal{V} \\
\xrightarrow{g} \mathcal{V}
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{V} \\
\xleftarrow{f} \mathcal{V} \\
\xleftarrow{g} \mathcal{V}
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{V} \\
\xleftarrow{f} \mathcal{V} \\
\xleftarrow{g} \mathcal{V}
\end{array},$$

where $s \in \mathfrak{gl}^1(\mathcal{V}) = \text{Hom}^1(\mathcal{V})/\beta'(\text{Hom}^2(\mathcal{V}))$, with:

$$g = f + \beta(s) = f + \partial s + s\partial.$$
The 2-category $\text{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

The vertical composition of 2-morphisms is

$$f + \beta(s) + \beta(t)$$

$$\uparrow(f + \beta(s), t)$$

$$\Rightarrow f + \beta(s)$$

$$\uparrow(f, s)$$

$$\Rightarrow f$$

$$\Rightarrow f + \beta(s) + \beta(t)$$

$$\uparrow(f, s + t)$$

$$\Rightarrow f$$
The 2-category $\text{Aut}(\mathcal{V})$ for a chain complex $\mathcal{V}$

The horizontal composition of 2-morphisms is

$$
\begin{align*}
&\begin{array}{c}
\uparrow (f_1, s) \\
\downarrow f_1
\end{array}
\quad \begin{array}{c}
\uparrow (g_1, t) \\
\downarrow g_1
\end{array}
\quad =
\begin{array}{c}
\uparrow (f_1 g_1, s.t) \\
\downarrow f_1 g_1
\end{array}
\end{align*}
$$

Here

$$
f_2 g_2 = f_1 g_1 + \beta (f_1 t + sg_2) = f_1 g_1 + \beta (sg_1 + f_2 t)
$$

and

$$
s.t = f_1 t + sg_2 = sg_1 + f_2 t.
$$

These coincide in

$$
\mathfrak{gl}^1(\mathcal{V}) = \frac{\text{Hom}^1(\mathcal{V})}{\beta' (\text{Hom}^2(\mathcal{V}))}.
$$
Two dimensional holonomy

- A path $x \xrightarrow{\gamma} y$ in a manifold $M$ is a piecewise smooth map $\gamma : [0, 1] \rightarrow M$, connecting $x$ and $y$.
- Given paths $x \xrightarrow{\gamma} y$ and $x \xrightarrow{\gamma'} y$ a 2-path $\gamma \xrightarrow{\Gamma} \gamma'$, written as:

  \[
  \xymatrix{ 
  x \ar@/_/[r]_{\gamma} & y \\
  x \ar@/^/[r]^{\gamma'} & y \ar@{=>}[u]^\Gamma
  }
  \]

  is given by piecewise smooth map $\Gamma : [0, 1]^2 \rightarrow M$, defining a homotopy $\gamma \rightarrow \gamma'$, relative to the boundary.
- These compose vertically and horizontally in the obvious way.
The 2-dimensional holonomy of a local 2-connection

Let $\mathcal{V}$ be a chain-complex of vector spaces, $M$ be a manifold, and $(A, B)$ a $\text{gl}(\mathcal{V})$-valued local 2-connection. There exists a 2-dimensional holonomy $\Gamma \mapsto \text{Hol}(\Gamma)$, which for a 2-path $\gamma \xrightarrow{\Gamma} \gamma'$ associates a 2-morphism $\text{Hol}(\Gamma)$ of $\text{Aut}(\mathcal{V})$, say:

$$
\begin{pmatrix}
\gamma' \\
\gamma
\end{pmatrix}
\begin{pmatrix}
\gamma
\
\Gamma
\
\gamma'
\end{pmatrix}

\xrightarrow{
\begin{pmatrix}
\gamma' \\
\gamma
\end{pmatrix}
\begin{pmatrix}
\gamma
\
\Gamma
\
\gamma'
\end{pmatrix}

= 

\begin{pmatrix}
\text{Hol}^1(\gamma') \\
\text{Hol}^2(\Gamma)
\end{pmatrix}
\begin{pmatrix}
\text{Hol}^1(\gamma) \\
\text{Hol}^2(\Gamma)
\end{pmatrix}

\xrightarrow{
\begin{pmatrix}
\text{Hol}^1(\gamma') \\
\text{Hol}^2(\Gamma)
\end{pmatrix}
\begin{pmatrix}
\text{Hol}^1(\gamma) \\
\text{Hol}^2(\Gamma)
\end{pmatrix}

$$

which preserves horizontal and vertical composites:

$$\text{Hol}(\Gamma\Gamma') = \text{Hol}(\Gamma)\text{Hol}(\Gamma')$$

and

$$\text{Hol} \left( \Gamma'' \right) = \frac{\text{Hol}(\Gamma)}{\text{Hol}(\Gamma')}.$$
As is the case of 1-dimensional holonomy, the variation of the holonomy when we vary the 2-paths is ruled by the 2-curvature 3-form:

**Theorem**

Suppose \((A, B)\) has zero 2-curvature 3-tensor \(\mathcal{M}_{(A,B)} = dB + A \wedge B\) and that \(\Gamma\) and \(\Gamma'\) are homotopic, relative to the boundary of \(D^2\). Then \(\text{Hol}^2(\Gamma) = \text{Hol}^2(\Gamma')\).
Let $\mathcal{V}$ be a chain complex of vector spaces. Recall the construction of the differential crossed module

$$\mathfrak{gl}(\mathcal{V}) = (\beta : \mathfrak{gl}^1(\mathcal{V}) \to \mathfrak{gl}^0(\mathcal{V}), \triangleright).$$

We are interested in 2-flat local 2-connections $(A, B)$ in $\mathbb{C}(n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : i \neq j \implies z_i \neq z_j \}$, with values in the differential crossed module $\mathfrak{gl}(\mathcal{V})$.

We thus want to define a $\mathfrak{gl}^0(\mathcal{V})$-valued 1-form $A$ and a $\mathfrak{gl}^1(\mathcal{V})$-valued 2-form $B$ such that:

$$\beta(B) = F_A = dA + \frac{1}{2} A \wedge A.$$

The 1-form $A$ should resemble the Knizhnik-Zamolodchikov connection $\sum_{i<j} \omega_{ij} r_{ij}$ with

$$\omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j}.$$
Consider a family of chain maps \( \{ r_{ab} \} \in \mathfrak{gl}_0(V) \) \((a, b \in \{1, \ldots, n\}, a \neq b)\) such that:

\[
    r_{ab} = r_{ba}, \quad [r_{ab}, r_{cd}] = 0 \quad \text{for} \quad \{a, b\} \cap \{c, d\} = \emptyset.
\]

Define a \(\mathfrak{gl}_0(V)\)-valued connection 1-form \(A\) over \(\mathbb{C}(n)\) as

\[
    A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab} \quad \text{where} \quad \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}.
\]

Put

\[
    R_{abc} = [r_{ab} + r_{ac}, r_{bc}]
\]

The curvature \(F_A = dA + \frac{1}{2}A \wedge A\) of \(A\) is then:

\[
    F_A = \sum_{1 \leq a < b < c \leq n} R_{bac} \omega_{ba} \wedge \omega_{ac} + R_{abc} \omega_{ab} \wedge \omega_{bc}.
\]
Flatness conditions for the 2-Knizhnik-Zamolodchikov connection

We now need a $\mathfrak{gl}^1(V)$-valued 2-form $B$ such that $\beta(B) = F_A$. We define a $\mathfrak{gl}^1(V)$-valued 2-form $B$ as:

$$B = \sum_{1 \leq a < b < c \leq n} K_{bac} \omega_{ab} \wedge \omega_{ac} + K_{abc} \omega_{ab} \wedge \omega_{bc}.$$ 

We must choose homotopies $K_{abc}, K_{bac} \in \mathfrak{gl}^1(V)$ (up to 2-fold homotopy), where $1 \leq a < b < c \leq n$, such that:

1. $\beta(K_{abc}) = R_{abc} = [r_{ab} + r_{ac}, r_{bc}]$ and $\beta(K_{bac}) = R_{bac}$.
2. $r_{ab} \triangleright K_{ijk} = 0$ if $\{a, b\} \cap \{i, j, k\} = \emptyset$. 
The 2-curvature 3-form $\mathcal{M}_{(A,B)} = dB + A \wedge B$ of $(A, B)$ vanishes, if and only if, the following conditions are satisfied:

\[
\begin{align*}
  r_{ad} \triangleright (K_{bac} + K_{bcd}) + (r_{ab} + r_{bc} + r_{bd}) \triangleright K_{cad} - (r_{ac} + r_{cd}) &\triangleright K_{bad} = 0 \\
  r_{bd} \triangleright (K_{abc} + K_{acd}) + (r_{ab} + r_{ad} + r_{ac}) &\triangleright K_{cbd} - (r_{bc} + r_{cd}) \triangleright K_{abd} = 0 \\
  r_{bc} \triangleright (K_{bad} + K_{cad}) + r_{ad} &\triangleright (K_{cbd} + K_{bcd} - K_{abc}) = 0 \\
  r_{ac} \triangleright (K_{abd} + K_{cbd}) + r_{bd} &\triangleright (K_{cad} + K_{acd} - K_{bac}) = 0 \\
  r_{cd} \triangleright (K_{bac} + K_{bad}) + (r_{ab} + r_{bc} + r_{bd}) &\triangleright K_{acd} - (r_{ac} + r_{ad}) \triangleright K_{bcd} = 0 \\
  r_{cd} \triangleright (K_{abc} + K_{abd}) + (r_{ab} + r_{ac} + r_{ad}) &\triangleright K_{bcd} - (r_{bd} + r_{bc}) \triangleright K_{acd} = 0
\end{align*}
\]

with $a < b < c < d \in \{1, \ldots, n\}$.

**Observation:** These relations are satisfied (in $\mathfrak{gl}^0(\mathcal{V})$) if

$$K_{abc} = [r_{ab} + r_{ac}, r_{bc}].$$

This follows from Bianchi identity $dF_A + A \wedge F_A = 0$. 

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joint with:

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Background

The Knizhnik-Zamolodchikov connection

Crossed modules and differential crossed modules

Complexes of vector spaces and differential crossed modules
Consider a representation $\sigma \mapsto \rho_\sigma$ of $S_n$ on $V$ by chain-complex maps. Choose chain complex maps

$$r_{ab} \in \mathfrak{gl}_0(V),$$

where $a, b \in \{1, \ldots, n\}$, with $a \neq b$, and also chain-homotopies

$$K_{ijk} \in \mathfrak{gl}^1(V),$$

where $i, j, k$ are distinct indices in $\{1, \ldots, n\}$. Satisfying

- $r_{ab} = r_{ba}$.
- $[r_{ab}, r_{cd}] = 0$ for $\{a, b\} \cap \{c, d\} = \emptyset$.
- $\beta(K_{ijk}) = R_{ijk} = [r_{ij} + r_{ik}, r_{jk}]$.
- $r_{ab} \triangleright K_{ijk} = 0$ if $\{a, b\} \cap \{i, j, k\} = \emptyset$.

We want that the two dimensional holonomy of $(A, B)$ descend to a two-dimensional holonomy in $\mathbb{C}(n)/S_n$. Therefore we now impose that for each $\sigma \in S_n$:

$$\rho_\sigma^{-1}(\sigma^*(A)) = A \text{ and } \rho_\sigma^{-1}(\sigma^*(B)) = B.$$
Flatness and $S_n$-equivariance conditions for the 2-Knizhnik-Zamolodchikov connection

Theorem (Cirio, FM)

The $\mathfrak{gl}(\mathbb{V})$-valued 2-connection $(A, B)$, where

\[
A = \sum_{a<b} \omega_{ab} r_{ab} \quad \text{and} \quad B = \sum_{a<b<c} K_{bac} \omega_{ab} \wedge \omega_{ac} + K_{abc} \omega_{ab} \wedge \omega_{bc}
\]

is invariant under the action of $S_n$, if, and only if:

\[
K_{abc} + K_{bca} + K_{cab} = 0, \quad K_{bca} = K_{bac}
\]

for each distinct $a, b, c, d \in \{1, \ldots, n\}$, and if for each $\sigma \in S_n$:

\[
r_{\sigma(a)\sigma(b)} = \rho_{\sigma}(r_{ab}) \quad \text{and} \quad K_{\sigma(a)\sigma(b)\sigma(c)} = \rho_{\sigma}(K_{abc}).
\]

Moreover, in such a case $(A, B)$ is 2-flat if, and only if,

\[
r_{ad} \triangleright (K_{bac} + K_{bcd}) + (r_{ab} + r_{bc} + r_{bd}) \triangleright K_{cad} - (r_{ac} + r_{cd}) \triangleright K_{bad} = 0,
\]

\[
r_{bc} \triangleright (K_{bad} + K_{cad}) - r_{ad} \triangleright (K_{dbc} + K_{abc}) = 0.
\]
The differential crossed module of horizontal 2-chord diagrams

\[ 2\text{ch}_n = (\beta : \text{ch}_n \to \text{ch}_n^+) \]

is the differential crossed module formally generated by the elements

\[ r_{ab} \in \text{ch}_n^+ \quad \text{and} \quad K_{abc} \in \text{ch}_n, \]

where \( a \neq b, a \neq c, b \neq c \), with relations:

- \( r_{ab} = r_{ba} \).
- \( [r_{ab}, r_{cd}] = 0 \) for \( \{a, b\} \cap \{c, d\} = \emptyset \).
- \( \beta(K_{abc}) = [r_{ab} + r_{ac}, r_{bc}] \).
- \( r_{ab} \triangleright K_{ijk} = 0 \) if \( \{a, b\} \cap \{i, j, k\} = \emptyset \).
- \( r_{ad} \triangleright (K_{bac} + K_{bcd}) + (r_{ab} + r_{bc} + r_{bd}) \triangleright K_{cad} - (r_{ac} + r_{cd}) \triangleright K_{bad} = 0 \).
- \( r_{bc} \triangleright (K_{bad} + K_{cad}) - r_{ad} \triangleright (K_{dbc} + K_{abc}) = 0 \).
- \( K_{abc} + K_{bca} + K_{cab} = 0 \).
- \( K_{bca} = K_{bac} \).
Infinitesimal R-matrices

An infinitesimal R-matrix in a Lie algebra $\mathfrak{g}$ is a symmetric tensor $r = \sum_i s_i \otimes t_i \in \mathfrak{g} \otimes \mathfrak{g}$, with

$$[r_{12} + r_{13}, r_{23}] = 0.$$ 

Given an infinitesimal $R$-matrix and a representation $V$ of $\mathfrak{g}$, the $\text{Hom}(V^\otimes, V^\otimes)$-valued connection 1-form in $\mathbb{C}(n)$:

$$A = \sum_{a < b} \phi_{ab}(r) \omega_{ab}, \text{ where } \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}$$

and

$$\phi_{ab}(r)(v_1 \otimes \ldots \otimes v_a \otimes \ldots \otimes v_b \otimes \ldots \otimes v_n)$$

$$= \sum_i v_1 \otimes \ldots \otimes s_i \triangleright v_a \otimes \ldots \otimes t_i \triangleright v_b \otimes \ldots \otimes v_n,$$

is flat and $S_n$-invariant.
Let $\mathcal{G} = (\partial : \mathfrak{h} \rightarrow \mathfrak{g}, \triangleright)$ be a differential crossed module. Define $\bar{\mathcal{U}}^{(n)}$ as being

$$\begin{align*}
\left( \mathfrak{h} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g} \right) & \oplus \left( \mathfrak{g} \otimes \mathfrak{h} \otimes \ldots \otimes \mathfrak{g} \right) \oplus \cdots \oplus \left( \mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{h} \right) \\
& \ldots \otimes \partial(v) \otimes \ldots \otimes w \otimes \cdots = \ldots \otimes v \otimes \ldots \otimes \partial(w) \otimes \ldots
\end{align*}$$

Example:

$$\mathcal{U}^{(3)} = \left( \mathfrak{h} \otimes \mathfrak{g} \otimes \mathfrak{g} \right) \oplus \left( \mathfrak{g} \otimes \mathfrak{h} \otimes \mathfrak{g} \right) \oplus \left( \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{h} \right)$$

\[
\begin{cases}
\partial(v) \otimes w \otimes X = v \otimes \partial(w) \otimes X \\
\partial(v) \otimes X \otimes w = v \otimes X \otimes \partial(w) \\
X \otimes \partial(v) \otimes w = X \otimes v \otimes \partial(w)
\end{cases}
\]

We define $\hat{\partial} : \mathcal{U}^{(3)} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ as:

$$\hat{\partial} = \partial \otimes \text{id} \otimes \text{id} + \text{id} \otimes \partial \otimes \text{id} + \text{id} \otimes \text{id} \otimes \partial.$$
Definition (free (and totally symmetric) infinitesimal 2R-matrix)

Let $\mathcal{G} = (\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be a differential crossed module. A (free) infinitesimal 2R-matrix in $\mathcal{G}$ is given by a symmetric tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$, and an element $P \in \bar{\mathcal{U}}^{(3)}$ such that, in $\bar{\mathcal{U}}^{(4)}$:

$$\hat{\partial}(P) = [r_{12} + r_{13}, r_{23}],$$

$$r_{14} \triangleright (P_{213} + P_{234}) + (r_{12} + r_{23} + r_{24}) \triangleright P_{314} - (r_{13} + r_{34}) \triangleright P_{214} = 0,$$

$$r_{23} \triangleright (P_{214} + P_{314}) - r_{14} \triangleright (P_{423} + P_{123}) = 0,$$

$$P_{123} + P_{231} + P_{312} = 0$$

$$P_{123} = P_{132}.$$

If these relations are satisfied only after applying a categorical representation $\rho$ of $(\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$, then $(r, P)$ is said to be an infinitesimal 2-R-matrix in $(\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$, with respect to $\rho$. 
Example Consider a Lie algebra \( g \), and the crossed module given by the the identity map \( g \xrightarrow{id} g \) and the adjoint action of \( g \) on \( g \). Then \( \mathcal{U}(n) = g \otimes n \).

Given any tensor \( r \in g \otimes g \), the pair \( (r, [r_{12} + r_{13}, r_{23}]) \) is an infinitesimal 2R-matrix.
Theorem

Let \((r, P)\) be an infinitesimal 2R-matrix in the differential crossed module \(G = (\partial : \mathfrak{h} \rightarrow \mathfrak{g}, \triangleright)\), with respect to a categorical representation of \(G\) on a complex of vector spaces \(V\). Consider the \(\mathfrak{gl}(V \otimes^n)-\)valued 2-connection \((A, B)\) on the configuration space \(C(n)\), defined as:

\[
A = \sum_{a < b} \omega_{ab} \tilde{\phi}_{ab}(r).
\]

\[
B = \sum_{a < b < c} \omega_{ab} \wedge \omega_{ac} \tilde{\phi}_{bac}(P) + \omega_{ab} \wedge \omega_{bc} \tilde{\phi}_{abc}(P)
\]

Then \((A, B)\) is a flat 2-connection, invariant the action of the symmetric group \(S_n\). Therefore its \(\text{Aut}(V)\)-valued holonomy descends to a two dimensional holonomy in \(C(n)/S_n\).
Differential crossed modules $\mathcal{G} = (\partial: \mathfrak{h} \to \mathfrak{g}, \rhd)$ are classified, up to weak equivalence, by a Lie algebra cohomology class $k \in H^3(\mathfrak{k}, \mathcal{M})$, where the differential crossed module $\mathcal{G}$ sits inside the exact sequence of Lie algebras

$$\{0\} \to \mathcal{M} \to \mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\text{proj}} \mathfrak{k} \to \{0\},$$

with $\mathcal{M}$ abelian, and $\mathfrak{k}$ has an induced action on $\mathcal{M}$.

Given a differential crossed module $\mathcal{G}$, the associated cohomology class (the $k$-invariant) is denoted by $k(\mathcal{G})$, and we say that $\mathcal{G}$ geometrically realises $k$. 
The string Lie 2-algebra

- The string Lie-2-algebra is a differential crossed module $\mathbf{String}$ geometrically realizing the Lie algebra 3-cocycle $\omega: \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$ with:

$$\omega(X, Y, Z) = \langle [X, Y], Z \rangle.$$

- Note that $\mathbf{String}$ is well defined up to weak equivalence (but not up to isomorphism).

- A very explicit realization of $\mathbf{String}$ is due to Wagemann.
Wagemann’s realization of String

Let $W_1$ be the Lie algebra of vector fields in one variable $x$:

$$\left[ f(x) \frac{d}{dx}, g(x) \frac{d}{dx} \right] = (f \frac{dg}{dx} - \frac{df}{dx} g)(x) \frac{d}{dx}.$$

Identify $\mathfrak{sl}_2(\mathbb{C}) \subset W_1$ as the sub-Lie algebra generated by

$$e_{-1} = \frac{d}{dx}, \quad e_0 = x \frac{d}{dx}, \quad e_1 = x^2 \frac{d}{dx}.$$

So that the commutation relations are:

$$[e_0, e_{-1}] = -e_{-1}, \quad [e_{-1}, e_1] = 2e_0, \quad [e_0, e_1] = e_1.$$
Wagemann’s realization of String

- Let $F_0$ be the space of polynomials in the variable $x$,
- Let $F_1$ the space of formal one-forms $f(x)dx$, where $f(x) \in F_0$.
- We consider $F_0$ and $F_1$ to be abelian Lie algebras. They are both $W_1$-modules via the Lie derivative:

$$\left(f(x) \frac{d}{dx}\right) \triangleright g(x) = (fg')(x)$$

$$\left(f(x) \frac{d}{dx}\right) \triangleright (g(x)dx) = (fg' + f'g)(x)dx.$$

- Hence they are $\mathfrak{sl}_2(\mathbb{C})$-modules as well
- Consider the 2-cocycle $\alpha : \mathfrak{sl}_2(\mathbb{C}) \wedge \mathfrak{sl}_2(\mathbb{C}) \to F_1$, defined as, in the basis $\{e_{-1}, e_0, e_1\}$ of $\mathfrak{sl}_2(\mathbb{C})$:

$$\alpha(e_0, e_1) = -\alpha(e_1, e_0) = 2dx,$$

and zero otherwise.
Wagemann’s realization of String

The differential crossed module String has the form

\[(\partial : F_0 \to F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C}) \triangleright),\]

where:

- if \((a, y), (b, z) \in F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C})\) we have:
  \[[(a, y), (b, z)] := (y \triangleright b - z \triangleright a + \alpha(y, z), [y, z]).\]
- \(\partial = (d, 0)\), where \(d\) denotes the formal de Rham differential.
- The Lie algebra \(F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C})\) acts on \(F_0\) via the action of \(\mathfrak{sl}_2(\mathbb{C})\) in \(F_0\) and the projection \(\pi : F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})\).
- The string differential crossed module can be embedded into the exact sequence:
  \[
  \{0\} \to \mathbb{C} \xrightarrow{i} F_0 \xrightarrow{\partial} F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\pi} \mathfrak{sl}_2(\mathbb{C}) \to \{0\}.\]
An infinitesimal 2-R-matrix in String

Let find an infinitesimal 2-R-matrix \((\bar{r}, P)\) in String, with \((\pi \otimes \pi)(\bar{r}) = r\), where \(r \in \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})\) is the infinitesimal R-matrix in \(\mathfrak{sl}_2(\mathbb{C})\):

\[
    r = 2 \ e_{-1} \otimes e_1 + 2 \ e_1 \otimes e_{-1} - 4 \ e_0 \otimes e_0 .
\]

Let \(\bar{r}\) be the obvious lift of \(r\) to 
\(\left( F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C}) \right) \otimes \left( F_1 \rtimes_{\alpha} \mathfrak{sl}_2(\mathbb{C}) \right)\). Explicitly:

\[
    \bar{r} = 2 \ (0, e_1) \otimes (0, e_{-1}) + 2 \ (0, e_{-1}) \otimes (0, e_1) - 4 \ (0, e_0) \otimes (0, e_0) .
\]

It holds that \([r_{12} + r_{13}, r_{23}] = 0\). However

\[
    \frac{1}{16} \ [\bar{r}_{12} + \bar{r}_{13}, \bar{r}_{23}] =
    \begin{align*}
    & (0, e_{-1}) \otimes (0, e_0) \otimes (dx, 0) + (0, e_{-1}) \otimes (dx, 0) \otimes (0, e_0) + \nonumber \\
    & - (0, e_0) \otimes (dx, 0) \otimes (0, e_{-1}) - (0, e_0) \otimes (0, e_{-1}) \otimes (dx, 0) .
    \end{align*}
\]
An infinitesimal 2-R-matrix in String

Let $P$ be the obvious lift of $[\bar{r}_{12} + \bar{r}_{13}, \bar{r}_{23}]$ to $U^{(3)}$. Explicitely:

\[
\frac{1}{16} P = (0, e_{-1}) \otimes (0, e_0) \otimes x + (0, e_{-1}) \otimes x \otimes (0, e_0) \\
- (0, e_0) \otimes x \otimes (0, e_{-1}) - (0, e_0) \otimes (0, e_{-1}) \otimes x.
\]

**Theorem (Cirio, JFM)**

The pair $(\bar{r}, P)$ is an infinitesimal 2-R-matrix in $\text{String}$ with respect to its adjoint representation.

Therefore in the adjoint categorical representation of $\text{String}$:

\[
\beta(P) = [\bar{r}_{12} + \bar{r}_{13}, \bar{r}_{23}], \\
\bar{r}_{14} \triangleright (P_{213} + P_{234}) + (\bar{r}_{12} + \bar{r}_{23} + \bar{r}_{24}) \triangleright P_{314} - (\bar{r}_{13} + \bar{r}_{34}) \triangleright P_{214} = 0 \\
\bar{r}_{23} \triangleright (P_{214} + P_{314}) - \bar{r}_{14} \triangleright (P_{423} + P_{123}) = 0 \\
P_{123} + P_{231} + P_{312} = 0 \\
P_{123} = P_{132}.
\]
Consider a surface braid $b_1 \xrightarrow{S} b_2$ without branch points, connecting the braids $b_1$ and $b_2$. This has an associated map $S' : D^2 \to \mathbb{C}(n)/S_n$. By using Chen integrals we can therefore define a surface holonomy

$$H(b_1) \xrightarrow{H(S')} H(b_2),$$

where $H(b_1)$ and $H(b_2)$ are valued in the algebra of formal power series in the universal enveloping algebra $\mathcal{U}(\text{ch}_n^+)$, and $H(S')$ is valued in the algebra of formal power series in $\mathcal{U}(2\text{ch}_n)$. **Problem:** Extend this surface holonomy to the case when $S$ has branch points. This will require some form of regularisation since, in the general case, the associated map $S' : D^2 \setminus \{\text{branch points}\} \to \mathbb{C}(n)/S_n$ will not be defined in all of $D^2$, however having a very particular type of singularities.
Invariants of braid cobordisms?

Drawing by Rui Carpentier.
Invariants of braid cobordisms from the String Lie-2-algebra?

- Is the holonomy of the Knizhnik-Zamolodchikov 2-connection derived from the infinitesimal 2-R-matrix on the string Lie-2-algebra convergent (or can it be regularised) for braid-cobordisms with branch points.
- Does it yield an interesting invariant of braid cobordisms?
- The non-trivial part of the homology will essentially live in

\[ H_1(HOM(\text{String}, \text{String})) = \text{Hom}(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C}) \]
The $k$-invariant of the differential crossed module of 2-chord diagrams?

**Problem:** Describe the kernel $M_n$ of the boundary map
\[ \partial : 2\text{ch}_n \to \text{ch}^+_n \] in the differential crossed module
\[ 2\text{ch}_n = (\partial : 2\text{ch}_n \to \text{ch}^+_n) \] of totally symmetric horizontal 2-chord diagrams.

By construction the cokernel is the Lie algebra $\text{ch}_n$ of horizontal chord diagrams, generated by $r_{ab}$, where $1 \leq a < b \leq n$, subject to the infinitesimal braid relations
\[
\begin{align*}
    r_{ab} &= r_{ba}; & [r_{ab} + r_{ac}, r_{bc}] &= 0; & [r_{ab}, r_{a'b'}] &= 0 \text{ if } \{a, b\} \cap \{a', b'\} = \emptyset.
\end{align*}
\]

Address whether the associated cohomology class
\[
k(2\text{ch}_n) \in H^3(\text{ch}_n, M_n)
\]
is trivial or not.

Here $2\text{ch}_n = (\partial : 2\text{ch}_n \to \text{ch}^+_n)$ is embedded in the exact sequence:
\[
\{0\} \to M_n \xrightarrow{i} 2\text{ch}_n \xrightarrow{\partial} \text{ch}^+_n \xrightarrow{\text{proj}} \text{ch}_n \to \{0\}.
\]
A Hopf algebra crossed module of 2-chord diagrams in the 2-sphere?

**Problem:** Is it possible to define a Hopf algebra crossed module of 2-chord diagrams in the 2-sphere from the relations defining the differential crossed module $\mathcal{C}_n$ of horizontal 2-chord diagrams?

**Problem** Spaces of (general) 2-chord diagrams for any 2-manifold?

**Problem:** Categorification of the STU- and IHX- relations?
Geometric framework for infinitesimal 2-R matrices

**Problem:** As infinitesimal R-matrices in a Lie algebra come naturally from invariant non-degenerate symmetric bilinear forms, it would be important to find a simple geometric way to construct infinitesimal 2R-matrices.

**Problem:** The most interesting case is when the 1-dimensional holonomy for braids and the 2-dimensional holonomy for braided surfaces derived from the Knizhnik-Zamolodchikov 2-connection are differential crossed module 1- and 2-intertwiners. For this to hold we need to impose a refinement of the notion of an infinitesimal 2-R-matrix. This categorifies the relation

\[ [r, \Delta(a)] = 0, \forall a \in g \]

much stronger than the 4-term relation

\[ [r_{12} + r_{13}, r_{23}] = 0. \]
Problem: Can we define a braided monoidal 2-category from the holonomy of the 2-Knizhnik-Zamolodchikov connection, considering the differential crossed module of horizontal 2-chord diagrams.

Problem: Drinfeld 2-Associators? (Florian Schätz)
Categorifying the Knizhnik-Zamolodchikov connection

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