Some risk processes associated to the debt function of a loan with variable interest rates

In previous work (reference 1), a risk random variable was introduced as the norm in a suitable space, of a stochastic process associated to the debt function of a loan. Some of the properties of the risk random variable were studied. In particular, under the hypothesis of interest rate constant in time, a simple condition on the debt function of the loan was shown to be sufficient in order to have control on the distribution function of the risk of the loan. In the following, we present some generalizations of those previous results, under the hypothesis of having the evolution of interest rates modelled by a known stochastic process.

1. Introduction

In this work we consider that a loan is characterized by the time evolution of the amount in debt. This is achieved with a stochastic process \((D_t)_{t \in [0,P]}\) which will describe for us the time evolution of the debt. This process is supposed to have some properties of regularity which are specified in the following theorems. We can associate with the debt process in a most natural way, a risk process. The risk associated with the loan will depend on a whole set of circumstances related to the financial situation of the borrower and to the general economic environment. As there is an uncertainty over these circumstances, the risk process is taken to depend on an uncertainty process \((\phi_t)_{t \in [0,P]}\). As general hypotheses on this process, one can consider it, to be a positive nondecreasing process with some regularity. If the trajectories of the risk process are in some vectorial normed space (e.g. the Lebesgue space \(L^2([0,P],B([0,P]),dx)\)) then, the random variable defined by the norm of the trajectories is the correspondent risk of the loan. In the context of variable interest rates the \(L^2\) risk is a natural concept.

In the following we suppose that in a certain complete probability space \((\Omega,\mathcal{A},\mathbb{P})\) we have a continuous version of a brownian process \((B_t)_{t \in [0,\infty]}\), which is a gaussian process uniquely determined in law by its mean \(\mathbb{E}[B_t] = 0\) and its covariance function \(\mathbb{E}[B_t B_s] = t \wedge s\). In the same probability space, the filtration naturally associated with this process is assumed to be given. For all details see reference 3 (page 47 on).

2. The debt process as a local martingale

An informal statement of our assumptions on the behavior of the risk process of a loan is that, the variation of the risk is proportional to the amount in debt, multiplied by the variation of this amount. The ratio of the proportion is given by the uncertainty process. These assumptions are summarized in the following expression,

\[
dR_t^\phi = \phi_t \, dD_t \, dD_t, \quad R_0^\phi = 0, \tag{1}
\]

or in integrated form:

\[
R_t^\phi = \int_0^t \phi_s \, dD_s \, dD_s. \tag{2}
\]

**Theorem 1.** Let the debt process \((D_t)_{t \in [0,P]}\) be a local martingale with respect to the brownian standard filtration and the uncertainty process \((\phi_t)_{t \in [0,P]}\) be a continuous adapted process with respect to the same filtration. Then, a sufficient condition for having the \(L^2\) risk verifying

\[
\mathbb{P}[\|R_t^\phi\|_2 > \alpha] \leq \varepsilon,
\]
is that:

\[
\left( \int_0^P (P - s) \mathbb{E}[\phi_s^2 D_s^2 G_s^2] \, ds \right)^{\frac{1}{2}} \leq 2 \alpha \varepsilon,
\]

where \( G_s \) is the process given by the Radon-Nikodym derivative of the mutual variation process \(< D_s, B_s >\), with respect to the Lebesgue measure.

Proof. The hypotheses made on the debt process entail that, this process has the representation given by:

\[
D_t = D_0 + \int_0^t G_s \, dB_s.
\]

As a consequence, the \( L^2 \) risk of the loan exists and, the following estimates for the mean of this random variable holds:

\[
\left( \mathbb{E}[||R_t^\phi||_2] \right)^{\frac{1}{2}} \leq \int_0^P \mathbb{E}[|| \int_0^t \phi_s D_s G_s \, dB_s ||^2] \, dt.
\]

Now, by the property of the isometry of the stochastic integral and by Fubini theorem, this gives:

\[
\left( \mathbb{E}[||R_t^\phi||_2] \right)^{\frac{1}{2}} \leq \int_0^P (P - s) \mathbb{E}[\phi_s^2 D_s^2 G_s^2] \, ds.
\]

Finally, an application of Markov inequality gives the desired result. For details on this proof see reference 2.

3. The uncertainty process as a local martingale

In this section we suppose that the risk process associated with the loan is given by:

\[
R_t^\phi = \frac{1}{2} [D_t^2 \phi_t + \int_0^t \phi_t^2 \, dB_t^2].
\]

This form of the risk process is a straightforward consequence in a deterministic context, of the form used above in (1) or (2). In previous work (reference 1), this form of the risk process was used to derive similar results when constant interest rates were supposed to hold.

Theorem 2. Let the debt process \((D_t)_{t \in [0, T]}\) be a continuous adapted process and the uncertainty process \((\phi_t)_{t \in [0, T]}\) be a local martingale with respect to the Brownian standard filtration. A sufficient condition for having the \( L^2 \) risk verifying

\[
\mathbb{P}[||R_t^\phi||_2 > \alpha] \leq \varepsilon,
\]

is that:

\[
\left( \int_0^P \mathbb{E}[D_t^2 \phi_t^2] \, ds \right)^{\frac{1}{2}} + \left( \int_0^P s \mathbb{E}[D_t^4 H_t^2] \, ds \right)^{\frac{1}{2}} \leq 2 \alpha \varepsilon,
\]

where \( H_s \) is the process given by the Radon-Nikodym derivative of the mutual variation process \(< \phi_s, B_s >\), with respect to the Lebesgue measure.

Remark 1. If \( \phi_t = F(B_t) \) where \( F \) is harmonic and \( F(0) = 0 \) then, by Itô’s formula, \( H_t \) is given by \( F'(B_t) \).

Proof. The proof of this result follows the same line of reasoning of theorem 1. For details see reference 2.

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4. References


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