Spot/Futures coupled model for commodity pricing\textsuperscript{1}

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Abstract

We propose, study and show how to price with a model for the coupled evolution of spot and future prices.

1. Introduction and motivation

1.1. On futures and forwards contracts

A futures contract at date $T$ is an agreement between two parties, transacted through a futures exchange such as the Chicago Board of Trade, CBOT or the New York Mercantile Exchange (NYMEX), struck at date zero to exchange at date $T$ a given quantity of a commodity, for a certain price. A forward contract at date $T$ is essentially the same as the future contract but is generally transacted over the counter.

An investor anticipating a price rise will buy futures. An investor anticipating a price decline will sell futures. Commodities are increasingly attractive to investors who view them as an alternative asset class justified by high liquidity, low transaction costs and no credit risk.

Important relationships between forward and futures prices must be stated. Under non-stochastic interest rates, and in absence of credit risk, forward and futures prices for the same underlying and maturity are equal. Also, for stochastic interest rates we may still suppose the equality as long as the covariance (under the pricing measure) between changes in the commodity price and interest rates is zero.

For futures prices modeling the main paradigm is a mean reverting model. In fact, until recently (2002) for most commodities it was believed, based on empirical studies, that commodity prices neither grow or decline on average over time; they tend to mean-revert to a level which may be viewed as the marginal

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cost of production see [1]. An important mean reverting model is given by the Ornstein-Uhlenbeck stochastic differential equation (SDE)
\[ dS_t = k(\theta - S_t)dt + \sigma dB_t \]
k, \theta > 0, (B_t)_{t \geq 0} is a brownian motion. In this model, \( S_t \) reverts on average to \( \theta \) at speed \( k \). One drawback of this model is that prices can be negative. An alternative model for futures prices modeling ensuring positivity mainly for energy and agricultural commodities is a model resembling geometric Brownian motion while having some form of mean reversion. This model is given by a mean reverting for the returns, that is, a geometric Ornstein-Uhlenbeck SDE.
\[ \frac{dS_t}{S_t} = k(\theta - \log(S_t))dt + \sigma dB_t \]
k, \theta > 0. By Ito's lemma \( \log(S_t) \) reverts on average to \( \theta - \sigma^2/2k \) at speed \( k \).

1.2. The convenience yield
The convenience yield is way of representing the advantages of ownership of physical goods introduced by Kaldor in [4] and Working in [7] and [8]. An informal definition of the convenience yield is the benefit that accrues to the owner of the physical commodity but not to the holder of a forward contract. That is, the convenience yield \( y \) is a rate such that, if \( S_t \) is the spot price, the benefit in dollar amount for the holder of the commodity over the interval \([t, t+dt] \) is \( S_t \times y \times dt \). An important result is that under no-arbitrage, constant interest rate \( r \) and constant convenience yield \( y \), if \( F^T_t \) represents (essentially) the futures price:
\[ F^T_t = S_t e^{(r-y)(T-t)} \].

1.3. The generalized convenience yield
The following facts highlights the state of present day commodity futures market. The volume transacted in the second fortnight of April in 2008 was \( 8.63 \cdot 10^6 \) tons and in 2007 was \( 5.58 \cdot 10^6 \) tons that is a 54% increase. In what concerns open interest, the number of outstanding contracts on a given day that have not been liquidated, in May 13, 2008 it was \( 2.9 \cdot 10^6 \) barrels and in December 13, 2007 it was \( 1.2 \cdot 10^6 \) barrels; which represents a 142 \% increase. The percentage of wheat trading in the US accounted for by investment funds was in January 2008 40 \% and in May 2008 60 \% ; that is a 50 \% increase in less than 5 months.

Data observation and these facts drove us to the following conclusions and the study proposal presented in this work. The actions of large speculators (large index funds, hedge funds) are causing a stronger connection between the value of a futures contract and the underlying value of the asset it is supposed to represent. To model this connection, we propose the following concept of a generalized convenience yield.
Definition 1. The generalized convenience yield process is the stochastic process \((y_t)_{t \geq 0}\) such that, \((r_t)_{t \geq 0}\) being the spot interest rate process, \((F^T_t)_{t \geq 0}\) the futures price process at maturity \(T\) and \((S_t)_{t \geq 0}\) the spot price process:

\[
F^T_t = S_t e^{(r_t - y_t)(T - t)}
\]

that is, by definition:

\[
y_t := r_t + \frac{1}{T - t} \ln \left( \frac{S_t}{F^T_t} \right)
\]

According to this definition we present in figure 1 oil and gold data as well as the corresponding generalized convenience yield minus the risk free instantaneous rate which we will call the storage yield.

**Figure 1:** One month futures and spot prices and the storage yield

Remark 1. Observe that as the generalized convenience yield approaches the spot interest rate, the spot price must approach the futures price. Conversely, if the spot price approaches the futures price the generalized convenience yield approaches the spot interest rate. This fact is observed in most of the commodities data studied until now.

1.4. The mathematical models

In order to reproduce the observed behavior of spot and futures process for these commodities we propose a diffusion model given by a system of SDE for joint
futures and spot evolution. The spot and futures prices are coupled by a system of SDE's,
\[
\begin{align*}
\frac{dS_t}{S_t} &= k^S(\theta^S - \log(F_t))dt + \sigma^S dB_t, \quad S_0 \in \mathbb{R}^+ \\
\frac{dF_t}{F_t} &= k^F(\theta^F - \log(S_t))dt + \sigma^F dB_t, \quad F_0 \in \mathbb{R}^+
\end{align*}
\]
(1)
where the process \((B_t)_{t \geq 0}\) is a unidimensional Brownian process. Notice that the parameters of this model \(k^S, k^F, \theta^S, \theta^F, \sigma^S,\) and \(\sigma^F\) are thought to depend on the maturity \(T\) of the futures contract.

By repeated use of Ito's formula, this system of SDE can be written as a multidimensional Ornstein-Uhlenbeck SDE.

\[
dZ_t = AZ_t dt + A\Sigma dB_t
\]
(2)
with \(Z_t = (Z^1_t, Z^2_t)^t, \Sigma = (\sigma^S, \sigma^F)^t,\) the prime denoting the transposed vector or matrix, and the matrix
\[
A = \begin{bmatrix} 0 & -k^S \\ -k^F & 0 \end{bmatrix}.
\]
The interpretation of equation (2) is of a vectorial Ito process which is treated in a more lengthy version of this work following [6]. By similarity with the unidimensional case we have the following solution.

**Proposition 1.** The process
\[
Z_t = e^{At}Z_0 + e^{At} \left( \int_0^t e^{-As} A\Sigma dB_s \right)
\]
(3)
is a solution for (2) SDE.

As a consequence we have the solution for the system (1) of SDE.

\[
S_t = \exp \left( \cosh(\sqrt{k^S k^F} t)(((k^S \theta^S - \sigma^S) - k^S \log(F_0)) - \\
- \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} t)((k^F \theta^F - \sigma^F) - k^F \log(S_0)) - \\
- \cosh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} s)k^F \sigma^S + \cosh(\sqrt{k^S k^F} s)k^S \sigma^F \right) dB_t + \\
+ \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^F}{k^S}} \sinh(\sqrt{k^S k^F} s)k^F \sigma^S + \cosh(\sqrt{k^S k^F} s)k^S \sigma^F \right) dB_t \right)
\]
(4)
and
\[
F_t = \exp \left( \cosh(\sqrt{k^S k^F} t)(((k^S \theta^S - \sigma^S) - k^S \log(F_0)) - \\
- \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} t)((k^F \theta^F - \sigma^F) - k^F \log(S_0)) - \\
- \cosh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} s)k^F \sigma^S + \cosh(\sqrt{k^S k^F} s)k^S \sigma^F \right) dB_t + \\
+ \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^F}{k^S}} \sinh(\sqrt{k^S k^F} s)k^F \sigma^S + \cosh(\sqrt{k^S k^F} s)k^S \sigma^F \right) dB_t \right)
\]
\[- \sqrt{\frac{k^F}{kS}} \sinh(\sqrt{k^S k^F} t) \left( (k^F \theta^F - \frac{\sigma^F}{2}) - k^F \log(S_0) \right) + \]
\[+ \sqrt{\frac{k^F}{kS}} \sinh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} s) k^F \sigma^S + \cosh(\sqrt{k^S k^F} s) k^S \sigma^F \right) dB_t - \]
\[- \cosh(\sqrt{k^S k^F} t) \int_0^t \left( \sqrt{\frac{k^F}{kS}} \sinh(\sqrt{k^S k^F} s) k^F \sigma^S + \cosh(\sqrt{k^S k^F} s) k^S \sigma^F \right) dB_t \right].

The results of model simulations, with the parameters estimated from the data for gold and oil, are presented in figure 2.

Figure 2: Simulated spot and one month futures prices and the storage yield

2. Pricing

Let us consider a classical market with two assets a risk free one \( R_t \) and some futures \( F_t \) having the following evolution laws.

\[
\begin{align*}
\frac{dR_t}{R_t} &= rR_t dt \\
\frac{dF_t}{F_t} &= \alpha(t, \omega) F_t dt + \sigma F_t dB_t
\end{align*}
\]

with \( \alpha(t, \omega) = k^F (\theta^F - \ln(S_t(\omega))) \) where \( S_t \) is the spot price. According to a classical result (see [5]), we have that the market is arbitrage free and complete if
and only if the following conditions is verified:

$$I_c := \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T \alpha(t, \omega) - r \frac{(\sigma_F)^2}{(k_F)^2} \, dt \right) \right] < +\infty$$

(5)

If this condition is verified then we may use the usual Black-Scholes formula as the instantaneous interest rate \( r \) and the volatility are constant. We may particularize condition (5) by using the formula (4) that gives us \( S_t \) explicitly. Writing that \( \ln(S_t) \) is Gaussian with variance \( \Psi(t) \) a straightforward computation yields that condition (5) is equivalent to

$$\int_0^T \Psi(t) \, dt < \frac{(\sigma_F)^2}{(k_F)^2}$$

In the practical cases studied this condition is verified for \( T < 10 \) days for oil and \( T < 25 \) days for gold. This verification was performed using the estimated values for the model parameters.

3. Model Estimation

The estimation was performed using quasi-likelihood estimation as detailed in [3, p. 122]. The main idea is to consider a Euler discretization of each of the SDE’s and observe that the transition density of \( \mathbb{E}[S_{t+\Delta t} \mid S_t = x] \) may be written explicitly. The method was implemented using Mathematica obtaining a robust estimation for \( \sigma_S \) and \( \sigma_F \).

<table>
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<tr>
<th></th>
<th>( k_S )</th>
<th>( \theta_S )</th>
<th>( \sigma_S )</th>
<th>( k_F )</th>
<th>( \theta_F )</th>
<th>( \sigma_F )</th>
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<td>0.2062</td>
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<td>6.78097</td>
<td>0.19847</td>
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</table>

4. Conclusions and further work

As to model results interpretation the main ideas are the following. The model proposed achieves a good replication of the overall behavior of the coupling between spot and futures prices. Nevertheless, the estimated generalized convenience yield, for both gold and oil, does not show a tendency to decrease. But we observe that the estimated long-term returns \( \theta_F \) and \( \theta_S \) are similar both for gold and oil, thus confirming an asymptotic connection between the returns of the futures and of the spot prices.

As to what concerns further work in the model exploration some ideas are the following. It is important to study of asymptotic properties of the model e.g., existence of invariant measure, the development of more efficient methods for calibration and estimation of the model and the benchmarking of the model against market prices of options given by institutional traders.
References


