RANDOM TEMPERED DISTRIBUTIONS ON (LOCALLY) COMPACT ABELIAN GROUPS

MLE

Abstract. We try to state the problem in form of questions, some of which, to be answered need only known results.

1. Some General Comments

It is well known [7][p. 268, 269] that, for a compact abelian group $G$, the Fourier inversion formula for $f \in L^2(\hat{G})$ takes the form:

$$f(x) = \sum_{\chi \in \hat{G}} <\chi, x> \hat{f}(\chi),$$

where $\hat{f}(\chi)$ is non zero only on a countable set of elements $\chi \in \hat{G}$.

Consider now, a family $(a_{\chi})_{\chi \in \hat{G}}$ which is square summable, that is, such that

$$\sum_{\chi \in \hat{G}} |a_{\chi}|^2 < +\infty .$$

In that case, $a_{\chi} \neq 0$ on a countable set of elements $\chi \in \hat{G}$ and so, $t$, defined on $G$ by:

$$t(x) = \sum_{\chi \in \hat{G}} a_{\chi} <\chi, x>$$

is a $L^2$ function, as the characters form a complete orthonormal system in $L^2(\hat{G})$. This harmonic analysis analogy, between the torus $\mathbb{T}$ and a general compact abelian group, allows us to define, at least formally, both test functions on the group and, by duality, tempered distributions on the group in the following way.

**Definition 1.1.** A test function $\phi$ on the group $G$ is a function defined by:

$$\phi(x) = \sum_{\chi \in \hat{G}} a_{\chi} <\chi, x> ,$$

where the family $(a_{\chi})_{\chi \in \hat{G}}$ is rapidly decreasing in a generalized sense associated with the notion of summability.

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*Date:* Informal report, August 20, 2003.

*1991 Mathematics Subject Classification.* Primary —–, —–; Secondary —–, ——.

*Key words and phrases.* Harmonic Analysis on (Locally) Compact Abelian Groups, Random Tempered Distributions.
Definition 1.2. $T$ is a tempered distribution on the group $G$ if there is a family $(b_{\chi})_{\chi \in \hat{G}}$, which is slowly increasing in a generalized sense associated with the notion of summability, such that:

$$T = \sum_{\chi \in \hat{G}} b_{\chi} \chi$$

meaning that, for a test function $\phi$ defined by $\phi(x) = \sum_{\chi \in \hat{G}} a_{\chi} < \chi, x >$:

$$<T, \phi> = \sum_{\chi \in \hat{G}} a_{\chi} b_{\chi} \chi.$$

2. Questions

Question 2.1. To define precisely the notions of rapidly decreasing and slowly increasing in a generalized sense associated with the notion of summability.

By analogy with the classical notion one can propose the following definition for a family of complex numbers to be rapidly decreasing.

Let $A = (a_i)_{i \in I}$ be a family of complex numbers indexed by an arbitrary set $I$. Denote by $S = S(A, J)$ the set of bijective maps from $J \subset I$, a finite set, to $\{1, \ldots, \#(J)\}$ such that for $\sigma \in S$:

$$|a_{\sigma^{-1}(1)}| \leq |a_{\sigma^{-1}(2)}| \leq \cdots \leq |a_{\sigma^{-1}(\#(J))}|.$$

Definition 2.2. The family $A = (a_i)_{i \in I}$ of complex numbers is rapidly decreasing in a generalized sense associated with the notion of summability if

$$\forall k \in \mathbb{N}^* \quad \sup_{J \subset I, \#(J) < +\infty} \sup_{\sigma \in S} \left( \sum_{n=1}^{\#(J)} |a_{\sigma^{-1}(n)}| n^k \right) < +\infty$$

Then, exactly in the same way:

Definition 2.3. The family $A = (a_i)_{i \in I}$ of complex numbers is slowly increasing in a generalized sense associated with the notion of summability if

$$\exists k \in \mathbb{N}^* \quad \sup_{J \subset I, \#(J) < +\infty} \sup_{\sigma \in S} \left( \sum_{n=1}^{\#(J)} \frac{|a_{\sigma^{-1}(n)}|}{n^k} \right) < +\infty$$

Question 2.4. Is a test function analytic in the sense of Mackey [10]? Would that have as a consequence that a test function is derivable in the sense of Bruhat [3]?

Question 2.5. Can one show that a tempered distribution is derivable in some sense? Is there an analog of the harmonic analysis on distributions on the torus for distributions on compact abelian groups?

Question 2.6. Can the result on the mean of random tempered distributions on the torus be stated and proved for random distributions on a compact group?

Question 2.7. What can be said for general locally compact abelian groups?

Question 2.8. Is there relevant applications for all this? Or, is there any particular case, besides the torus, that deserves particular study?
References

10. Mackey, Laplace Transforms for locally compact abelian groups, *Proceedings of the National Academy of Sciences of USA* 34 (1948), 156–.

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