ON THE SPACE OF RANDOM TEMPERED DISTRIBUTIONS HAVING A MEAN

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ABSTRACT. A class of random tempered distributions on $\mathbb{R}$ is introduced by considering random series in the usual Hermite functions having as coefficients random variables which satisfy certain growth conditions. This class is shown to be exactly the class of random Schwartz distributions having a mean. Otherwise stated, we obtain a characterization of the stochastic processes with a first moment and having as trajectories tempered distributions. As important examples of this class, we introduce a brownian type process on $\mathbb{R}$ and recall the brownian distributions of J.-P. Kahane. We present a study on a possible converse of a result on brownian distributions which leads to a moment problem.

1. INTRODUCTION

A natural starting point to study random tempered distributions is, to consider as a definition that, such an object is just a map from $\Omega$ into $\mathcal{S}'(\mathbb{R})$, the space of tempered or Schwartz distributions on $\mathbb{R}$, the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ being complete (see [14, p. 9] or [11, p. 13]).

A complementary point of view is, to define from the start, a random distribution as a linear continuous map from $\mathcal{S}(\mathbb{R})$, the Schwartz space of test functions, into some space of random variables, endowed with some topologi-

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cal (and hence measurable) structure, usually, convergence in probability (see [3, p. 210]).

Some price has to be paid for the simplicity of the first approach. In fact, as soon as some result is to be proved, requiring the map between $\Omega$ and $\mathcal{S}'(\mathbb{R})$ to be a random variable, the existence of the law of such a random variable being an instance, the topological structure of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ has to be considered (see [11, p. 115] or [12, p. 21]).

In this work, we adopt first the simplest approach in order to define and study a particular class of tempered random distributions. The characterization result (theorem 9) proved in the sequel, which amounts to show that this class recovers exactly the space of random tempered distributions having a mean \(^1\) uses only a trivial consequence of the measurability condition. Namely we use the fact that the sequence of Fourier Hermite coefficients of the random distribution is a sequence of random variables. Using the denominations of [11], the characterization of amenable random distributions can be read as a structure result on the generalized stochastic processes having as trajectories tempered distributions.

Results on the Minlos' support of the law of a generalized stochastic process having (generalized) moments are given in [3, p. 212]. In [1], the idea of considering gaussian random fields whose covariance is the parametric of an elliptic pseudo-differential operator, gives results on the sample path properties of such a gaussian field. This idea could perhaps be used, in a modified form, to obtain more information on the structure of generalized processes whose trajectories are tempered distributions.

2. THE HERMITE FUNCTIONS

2.1. Laurent Schwartz's fundamental result. In this introductory section we summarize from [19, p. 261], some definitions and results on the Hermite functions. The most important one is theorem 2, which establishes that the space of tempered distributions on $\mathbb{R}$ is isomorphic to the space of slowly increasing sequences indexed by the integers. The formula presented in lemma 1 will be used in the study of brownian distributions.

Definition 1. The Hermite polynomials on $\mathbb{R}$ are the polynomials $H_m(x)$, for $m \in \mathbb{N}$, defined by the equations:

$$
\frac{d^m}{dx^m} e^{-2\pi x^2} = (-1)^m \sqrt{m!} 2^{m-\frac{1}{2}} \pi^{\frac{m}{2}} H_m(x) e^{-2\pi x^2}.
$$

\(^1\)Amenable random distributions was the designation proposed by J.-P. Kahane for tempered random distributions having a mean.
Associated with these polynomials are the Hermite functions given, for \( m \in \mathbb{N} \), by:
\[
\mathcal{H}_m = H_m(x)e^{-\pi x^2}.
\]

**Proposition 1.** The family \((\mathcal{H}_m)_{m \in \mathbb{N}}\) is a complete orthonormal set of functions in \( L^2(\mathbb{R}) \).

The Fourier transform of an Hermite function is simply given by the following proposition.

**Proposition 2.**
\[
\mathcal{F}(\mathcal{H}_m)(\xi) = (-1)^m \mathcal{H}_m(\xi).
\]

As a consequence, we have the representation for the elements of \( L^2(\mathbb{R}) \), as a series of Hermite functions.

**Proposition 3.** For every \( \varphi \in L^2(\mathbb{R}) \), the following equality in the \( L^2 \) sense holds:
\[
\varphi(x) = \sum_{m=0}^{+\infty} a_m(\varphi) \mathcal{H}_m(x),
\]
where the coefficients \((a_m(\varphi))_{m \in \mathbb{N}}\) are given by:
\[
a_m(\varphi) = \int_{\mathbb{R}} \varphi(x) \mathcal{H}_m(x) \, dx
\]
and, the following Plancherel type result holds:
\[
\sum_{m=0}^{+\infty} |a_m(\varphi)|^2 = \int_{\mathbb{R}} |\varphi(x)|^2 \, dx.
\]

The natural operators for the Hermite functions are given by:

(2.1) \[
(\tau_+ f)(x) = (\frac{d}{dx} f)(x) + 2\pi x f(x) \quad (\tau_- f)(x) = -(-\frac{d}{dx} f)(x) + 2\pi x f(x) .
\]

where \( f \) is some regular function (or a distribution). The recurrence relations that hold among the Hermite function imply the next result about the action of \( \tau_+ \) and \( \tau_- \) on the Hermite functions.

**Proposition 4.** For every \( m \geq 1 \) or \( p \geq 0 \):

(2.2) \[
\tau_+(\mathcal{H}_m) = 2\sqrt{\pi m} \, \mathcal{H}_{m-1} , \quad \tau_-(\mathcal{H}_p) = 2\sqrt{\pi(p+1)} \, \mathcal{H}_{p+1}.
\]
As a consequence, if $\varphi$, $\varphi'$, $x\varphi$, are functions in $L^2(\mathbb{R})$ then, as $\tau_+$ and $\tau_-$ are adjoint operators of each other,

$$a_m(\tau_+\varphi) = \int_{\mathbb{R}} (\tau_+\varphi)(x) \mathcal{H}_m(x) \, dx = \int_{\mathbb{R}} \varphi(x) (\tau_+\mathcal{H}_m)(x) \, dx.$$ 

This relation obviously implies that:

$$a_m(\tau_+\varphi) = 2\sqrt{\pi(m+1)} \, a_{m+1}(\varphi).$$

And, by the same line of reasoning, we also have that, for $m \geq 1$:

$$a_m(\tau_-\varphi) = 2\sqrt{\pi m} \, a_{m-1}(\varphi),$$

with the natural convention that $a_0(\tau_-\varphi) = 0$. These formulae now imply the next proposition.

**Proposition 5.** The following conditions are equivalent.

1. $\varphi$, $\varphi'$, $x\varphi$, are in $L^2(\mathbb{R})$.
2. $\sum_{m=0}^{+\infty} |a_m(\varphi)|^2 < +\infty$.

By induction, using the iterates of the operators $\tau_+$ and $\tau_-$, we get the following theorem which gives a characterization of the space $S$ of rapidly decreasing functions.

**Theorem 1.** A necessary and sufficient condition for having $\varphi \in S$, is that the sequence $(a_m(\varphi))_{m \in \mathbb{N}}$ is a rapidly decreasing sequence. The natural map that gives the sequence $(a_m(\varphi))_{m \in \mathbb{N}}$ as a function of $\varphi$, is an isomorphism of topological vector spaces between the space $S$ and, the space of rapidly decreasing sequences.

Given a tempered distribution $T$, the Hermite coefficients $(a_m(T))_{m \in \mathbb{N}}$ of $T$, can naturally be calculated by:

$$a_m(T) = \langle T, \mathcal{H}_m \rangle,$$

as the functions $\mathcal{H}_m$ all belong to $S$. Using the representation of $T$ as a finite sum of distributions which in turn are images of functions in $L^2(\mathbb{R})$ by repeated applications of the operators $\tau_+$ and $\tau_-$, one can prove that the sequence $(a_m(T))_{m \in \mathbb{N}}$ is a slowly increasing sequence.

Conversely, given a slowly increasing sequence $(b_m)_{m \in \mathbb{N}}$, the series $\sum_{m=0}^{+\infty} b_m \mathcal{H}_m$ converges in $S'$ to a limit, let it be $T$, such that:

$$\forall m \in \mathbb{N} \quad b_m = a_m(T)$$

In short, we have the following important result.
Theorem 2 (Laurent Schwartz). A necessary and sufficient condition for having \( T \in S' \) is that the sequence \( (a_m(T))_{m \in \mathbb{N}} \) is a slowly increasing sequence. The natural map that associates the sequence of the Hermite coefficients of the distribution \( T \), to the distribution, is an isomorphism between \( S' \) and the space of slowly increasing sequences.

The representation of the duality between \( S \) an \( S' \), given the Hermite coefficients of \( T \in S' \) and \( \varphi \in S \), is expressed by:

\[
<T, \varphi> = \sum_{m=0}^{+\infty} a_m(T) a_m(\varphi).
\]

The expression of the Fourier transform of an object in \( S \) or in \( S' \) and, represented by a Hermite series:

\[
\sum_{m=0}^{+\infty} c_m \mathcal{H}_m,
\]

is given by:

\[
\mathcal{F}(\sum_{m=0}^{+\infty} c_m \mathcal{H}_m) = \sum_{m=0}^{+\infty} (-1)^m c_m \mathcal{H}_m,
\]

where the series converge in \( S \), if the sequence \( (c_m)_{m \in \mathbb{N}} \) is rapidly decreasing and the series converge in \( S' \), if the sequence is slowly increasing.

2.2. The generating function for the square of Hermite functions. The formula presented in the next lemma, will be instrumental in the proof of a theorem of J. P. Kahane on brownian distributions as well as, in our investigation about a converse of the proposition stated in that theorem.

**Lemma 1.** For \( |t| < 1 \):

\[
\sum_{m=0}^{+\infty} t^m \mathcal{H}_m^2(x) = \frac{2}{1-t^2} e^{-2\pi x^2 \frac{t^4}{1+t^4}}
\]

**Proof.** This formula, which we took without proof from [13], can be derived from a similar one already proved in [22, p. 77, 78]. In fact, theorem 43 of the monograph just quoted reads that, for \( |t| < 1 \):

\[
\sum_{n=0}^{+\infty} \frac{t^n}{2^n n!} (P_n(x))^2 = \frac{1}{\sqrt{1-t^2}} e^{-x^2 \frac{1-t^4}{1+t^4}},
\]

where the Hermite polynomials \( P_n(x) \) are defined as those polynomials which satisfy for \( n \in \mathbb{N} \):

\[
P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]
This formula shows that:
\[
\frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{-x^2} P_n(x).
\]
We observe that trivially:
\[
\frac{d^n}{dx^n} e^{-2\pi x^2} = (2\pi)^{\frac{n}{2}} \frac{d^n}{dy^n} (e^{-y^2})|_{y=\sqrt{2\pi} x}.
\]
After making the appropriate substitutions, we have the following relation between Titchmarsh Hermite polynomials \( P_n \) and the Schwartz Hermite polynomials \( H_n \):
\[
\forall n \in \mathbb{N} \quad P_n(\sqrt{2\pi} x) = 2^{\frac{2n-1}{4}} \sqrt{n!} H_n(x).
\]
Using this relation in formula 2.6, we get the result announced. \( \square \)

3. **Random Schwartz distributions having a mean**

**3.1. Introduction.** In this section we introduce a space of random Schwartz distributions over \( \mathbb{R} \) by means of series of Hermite functions having as coefficients random variables satisfying a certain growth condition. The results presented here, are similar to those presented in [4], where the particular case of periodic distributions was studied.

Let \( (\Omega, \mathcal{A}, \mathbb{P}) \) be a complete probability space. Let \( \mathcal{M} \) denote the space of random variables taking complex values and \( \mathcal{M}^\mathbb{N} \) the space of sequences of elements of \( \mathcal{M} \) indexed by the integers \( \mathbb{N} \). For \( A, \) an element of \( \mathcal{M} \), let \( \mathbb{E}[|A|] \) be defined by:
\[
\mathbb{E}[|A|] = \int_\Omega |A| \, d\mathbb{P}.
\]
Let
\[
C_m = \{(c_n)_{n \in \mathbb{N}} \in \mathcal{M}^\mathbb{N} : (\exists A \in \mathcal{M}, A \geq 0, \mathbb{E}[A] < +\infty) (\exists k \in \mathbb{N}) (\forall n \in \mathbb{N}) |c_n| \leq A(1 + n)^k \text{ a. s. on } \Omega \},
\]
be a space of sequences of random variables. This space can be described by equivalent conditions given in the next theorem.

**Theorem 3.** For \( (c_n)_{n \in \mathbb{N}} \in \mathcal{M}^\mathbb{N} \) the following are equivalent:

1. \( (c_n)_{n \in \mathbb{N}} \in C_m \)

2. There exists an integrable random variable \( A \) and a real bounded random variable \( K \), defined on \( \Omega \), such that:
\[
\forall n \in \mathbb{N} \quad |c_n| \leq A(1 + n)^K \text{ a. s. on } \Omega
\]
(3) The sequence \( (\mathbb{E}[|c_n|])_{n \in \mathbb{N}} \) is a sequence of slow growth or, in an equivalent rephrasing:
\[
(\exists a > 0) \ (\exists k \in \mathbb{N}) \ (\forall n \in \mathbb{N}) \quad \mathbb{E}[|c_n|] \leq a (1 + n)^k
\]

Proof. The proof goes exactly as in [4], replacing \( \mathbb{Z} \) by \( \mathbb{N} \) in every instance where the first set is the set of indices of the sequence. \( \Box \)

Associated with every sequence \( (c_n)_{n \in \mathbb{N}} \) in \( \mathcal{C}_m \), there is a random Schwartz distribution \( T \) on \( \mathbb{R} \), defined for a \( \varphi \in \mathcal{S} \) by:
\[
< T, \varphi > = \sum_{m=0}^{+\infty} c_m a_m(\varphi).
\]

The distribution \( T \) is well defined because by the second condition of the theorem 3, the sequence \( (c_n)_{n \in \mathbb{N}} \) is almost surely a sequence of slow growth. Besides, the sequence \( (a_m(\varphi))_{m \in \mathbb{N}} \) of the Hermite coefficients of \( \varphi \) is a rapidly decreasing sequence and so, the series in the left hand side of 3.2 converges almost surely.

We will present some examples after mentioning some properties of the random Schwartz distributions just defined.

Theorem 4 (On the unicity of the representation). Let \( T \) be given as in 3.2 by a sequence \( (c_n)_{n \in \mathbb{N}} \). Then the following are equivalent:

1. \( T = 0 \) a. s. on \( \Omega \).
2. \( \forall m \in \mathbb{N} \quad c_m = 0 \) a. s. on \( \Omega \).

Proof. Same proof as in the corresponding result in [4]. \( \Box \)

As a consequence, for every random Schwartz distribution defined this way there is, up to equality almost sure, an unique sequence of random variables such that formula 3.2 is verified almost surely. Such a sequence will be denoted by:
\[
(a_m(T))_{m \in \mathbb{N}}.
\]

An obvious corollary of this theorem gives a necessary and sufficient condition for equality of two random Schwartz distributions obtained by the way just explained.

Theorem 5. Let \( T, U \), be two random Schwartz distributions obtained by formula 3.2 from the sequences \( (a_m(T))_{m \in \mathbb{N}} \) and \( (a_m(U))_{m \in \mathbb{N}} \) respectively. Then the following are equivalent:

1. \( T = U \) a. s. on \( \Omega \).
2. \( \forall m \in \mathbb{N} \quad a_m(T) = a_m(U) \) a. s. on \( \Omega \).

One important property of the class of random tempered distributions here introduced by means of the space \( \mathcal{C}_m \) is that, this class is stable by derivation.
Definition 2. For a random Schwartz distribution $T$, the derivative $T'$ of $T$ is a random Schwartz distribution defined by:

\begin{equation}
\forall \omega \in \Omega \quad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad < T', \varphi > = - < T, \varphi > .
\end{equation}

If a random tempered distribution has the sequence of its Hermite coefficients given by 2.3, in the class $C_m$ then, the Hermite coefficients of the derivative are also in the class $C_m$. This fact follows from the next result.

Theorem 6. Let $T = \sum_{m=0}^{+\infty} a_m(T) \mathcal{H}_m$ be a random Schwartz distribution. Then, the derivative of $T$ has a representation as a Hermite series given by:

\begin{equation}
T' = a_1(T) \sqrt{\pi} \mathcal{H}_0 + \sum_{m=1}^{+\infty} \left( a_{m+1}(T) \sqrt{\pi(m+1)} - a_{m-1}(T) \sqrt{\pi m} \right) \mathcal{H}_m .
\end{equation}

Proof. As a consequence of the definition 3.3 and of duality formula 2.4 we have that, for any test function $\varphi \in \mathcal{S}$:

\begin{equation}
<T', \varphi > = < T, (-\varphi') > = \sum_{m=0}^{+\infty} a_m(T) (-a_m(\varphi')) .
\end{equation}

As a result of an integration by parts, the Hermite coefficients of $-\varphi'$ are given for $m \in \mathbb{N}$ by:

\begin{equation}
-a_m(\varphi') = \int_{\mathbb{R}} \varphi(u) \mathcal{H}'_m(u) \, du .
\end{equation}

Now, the derivative of any Hermite function can be simply obtained with the operators $\tau_+$ and $\tau_-$. In fact, it follows from the definition of these operators (2.1) that:

\begin{equation}
\frac{d}{dx} = \frac{1}{2} (\tau_+ - \tau_-) .
\end{equation}

Then, proposition 2.2 imply that for $m \geq 1$,

\begin{equation}
\mathcal{H}'_m = \sqrt{\pi m} \mathcal{H}_{m-1} - \sqrt{\pi(m+1)} \mathcal{H}_{m+1}
\end{equation}

and, as $\tau_+ (\mathcal{H}_0) = 0$ that, $\mathcal{H}'_0 = -\sqrt{\pi} \mathcal{H}_1$. These expressions for the derivatives of Hermite functions, together with formula 3.6 imply that for $m \geq 1$:

\begin{equation}
-a_m(\varphi') = a_{m-1}(\varphi) \sqrt{\pi m} - a_{m+1}(\varphi) \sqrt{\pi(m+1)}, \quad -a_0(\varphi') = -a_1(\varphi) \sqrt{\pi} .
\end{equation}

After replacing formula 3.7 in formula 3.5 we get:

\begin{equation}
<T', \varphi > = a_1(T) \sqrt{\pi} a_0(\varphi) + \sum_{p=1}^{+\infty} \left( a_{p+1}(T) \sqrt{\pi(p+1)} - a_{p-1}(T) \sqrt{\pi p} \right) a_p(\varphi) ,
\end{equation}
which is exactly the formula stated in the theorem \(\Box\)

Due to the hypotheses made on a sequence in \(C_m\), a random Schwartz distribution given by way of formula 3.2, has a mean in the sense of the next definition.

**Definition 3.** Let \(T\) be a random Schwartz distribution associated to a sequence \((a_m(T))_{m \in \mathbb{N}}\) by formula 3.2. Then, \(T\) admits \(\tilde{T}\) as a mean if and only if:

1. \(\forall \varphi \in \mathcal{S} \quad <T, \varphi> \in \mathcal{M} \cap L^1(\Omega)\).
2. \(\forall \varphi \in \mathcal{S} \quad \mathbb{E}[<T, \varphi>] = <\tilde{T}, \varphi>\).

A random Schwartz distribution built with a sequence \((c_m)_{m \in \mathbb{N}} \in C_m\) by formula 3.2 does admit a mean. This mean has a representation as a Hermite function series having as coefficients the sequence of expectations \((\mathbb{E}[c_m])_{m \in \mathbb{N}}\). This simple result follows from a common application of Lebesgue dominated convergence theorem as we will show next.

**Theorem 7.** If \(T = \sum_{m=0}^{+\infty} a_m(T) \mathcal{H}_m\) is a random Schwartz distribution then, \(T\) admits the usual Schwartz distribution \(\tilde{T} = \sum_{m=0}^{+\infty} \mathbb{E}[a_m(T)] \mathcal{H}_m\) as a mean.

**Proof.** For every \(\varphi \in \mathcal{S}\), the sequence \((a_m(\varphi))_{m \in \mathbb{N}}\) is a rapidly decreasing sequence. By theorem 3, the sequence \((\mathbb{E}[a_m(T)])_{m \in \mathbb{N}}\) is a slow growth sequence and so, the series:

\[
\sum_{m=0}^{+\infty} \mathbb{E}[|a_m(T)|] |a_m(\varphi)|,
\]

converges almost surely. Now, by the Lebesgue dominated convergence theorem:

\[
<T, \varphi> = \sum_{m=0}^{+\infty} \mathbb{E}[a_m(T)] a_m(\varphi) = \mathbb{E}[\sum_{m=0}^{+\infty} a_m(T) a_m(\varphi)] = \mathbb{E}[<T, \varphi>] ,
\]

where the last equality results from the duality formula 2.4. \(\Box\)
3.2. A characterization. A noticeable converse of theorem 7 holds in a sense that we now proceed to explain. Let $T$ be a measurable map from $(\Omega, \mathcal{A})$ into $\mathcal{S}'$, which we consider endowed with the Kolmogorov $\sigma$-algebra associated to the dual countably hilbertian (or Schwartz) topology on $\mathcal{S}$ (see [12, p. 6, 16]). Then, for almost every $\omega \in \Omega$ the Hermite coefficients of $T(\omega)$, which we denote as usual by

$$a_m(T(\omega)) = \langle T(\omega), \mathcal{H}_m \rangle,$$

for $m \in \mathbb{N}$, are all well defined and furthermore, we can consider $(a_m(T))_{m \in \mathbb{N}}$ as a well defined sequence of random variables. For a general $T$, no growth condition on the sequence $(a_m(T))_{m \in \mathbb{N}}$ is verified so as to ensure that this sequence is in the class $\mathcal{C}_m$. For instance, consider a sequence in which only a finite number of terms are non zero and having one of these as a non integrable random variable.

Another example is given by a sequence of random variables taking small values with a big probability, and big values with small probability such as, $(a_m(T))_{m \in \mathbb{N}^*}$ verifying:

$$\mathbb{P}[a_m(T) = me^m] = \frac{1}{m^2}, \quad \mathbb{P}[a_m(T) = 0] = 1 - \frac{1}{m^2}.$$

As the series $\sum_{m=1}^{+\infty} \mathbb{P}[a_m(T) = me^m]$ converges, then, by Borel-Cantelli lemma, $T$ is almost surely given by a finite sum of terms. That is, for almost all $\omega \in \Omega$ there exists a $N \in \mathbb{N}^*$, $N = N(\omega)$ such that:

$$T(\omega) = \sum_{m=1}^{N} a_m(T) \mathcal{H}_m.$$

Now, as we have:

$$\forall m \in \mathbb{N}^* \quad \mathbb{E}[a_m(T)] = \frac{e^m}{m},$$

which, does not define a sequence of slow growth, the sequence $(a_m(T))_{m \in \mathbb{N}^*}$ can not be in the class $\mathcal{C}_m$ as a result of theorem 3.

The next theorem shows that the first condition in definition 3 is a sufficient condition on $T$ for the sequence $(a_m(T))_{m \in \mathbb{N}}$ to be in the class $\mathcal{C}_m$.

**Theorem 8.** Let $T$ be defined (almost surely) in $\mathcal{S}$, by a sequence of random variables $(a_m(T))_{m \in \mathbb{N}}$ such that:

$$\forall \varphi \in \mathcal{S} \quad < T, \varphi > = \sum_{m=0}^{+\infty} a_m(T) a_m(\varphi) \text{ a. s. on } \omega.$$

Then, if:

$$\forall \varphi \in \mathcal{S} \quad < T, \varphi > \in L^1(\Omega),$$

...
the sequence \((a_m(T))_{m \in \mathbb{N}}\) is in the class \(C_m\).

Proof. The proof is a straightforward adaptation of the proof given to a similar result for periodic Schwartz distributions in [5]. Let us show first, using the closed graph theorem that, the map \(\Lambda_T\) defined for every test function \(\varphi \in \mathcal{S}(\mathbb{R})\) by \(\Lambda_T(\varphi) = \langle T, \varphi \rangle\), is continuous from \(\mathcal{S}(\mathbb{R})\) into \(L^1(\Omega)\). As \(T\) is, almost surely, a random tempered distribution, if \((\varphi_l)_{l \in \mathbb{N}}\) is a sequence of test functions converging to zero in \(\mathcal{S}(\mathbb{R})\), then the sequence of random variables \((U_l)_{l \in \mathbb{N}}\) defined for \(l \in \mathbb{N}\) by:

\[
U_l = \langle T, \varphi_l \rangle,
\]

converges a.s. to zero. So, this sequence \((U_l)_{l \in \mathbb{N}}\) converges also in probability to zero. Now, suppose that \((U_l)_{l \in \mathbb{N}}\) converges to \(U\) in \(L^1(\Omega)\); then, the sequence converges also in probability to \(U\) and so \(U = 0\). By the closed graph theorem [18, p. 51], the map \(\Lambda_T\) is continuous. Taking in account the topologies of \(\mathcal{S}(\mathbb{R})\) and \(L^1(\Omega)\), the continuity of \(\Lambda_T\) can be expressed in the following way:

\[
\exists k \in \mathbb{N}, \ \exists c_k > 0 \ \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \| \langle T, \varphi \rangle \|_{L^1(\Omega)} \leq c_k \sup_{n \in \mathbb{N}} (1 + |n|)^k |a_n(\varphi)|.
\]

As a second step, we use the sequence of Rademacher functions [25, p. 212] defined in \([0, 1]\) by:

\[
\forall t \in [0, 1] \quad \forall n \in \mathbb{N} \quad r_n(t) = \sigma(\sin(2^{n+1}\pi t)) \quad ,
\]

where \(\sigma(t)\) denotes the sign of \(t\) defined by:

\[
\sigma(t) = \begin{cases} 
\frac{|t|}{t} & \text{if } t \neq 0 \\
0 & \text{otherwise} 
\end{cases}.
\]

And, we also consider \(s(\mathbb{N})\), the space of rapidly decreasing complex sequences with the topology induced by the quasi-norms \(\| \cdot \|_k\), \(k \in \mathbb{N}\) which are defined for \(s = (s_n)_{n \in \mathbb{N}}\), an element of \(s(\mathbb{N})\), by:

\[
|s|_k = \sup_{n \in \mathbb{N}} (1 + n)^k |s_n|.
\]

Then, for every \(t \in [0, 1]\), the map from \(s(\mathbb{N})\) to \(s(\mathbb{N})\) which associates to each sequence \(s \in s(\mathbb{N})\), the sequence \(w = (w_n)_{n \in \mathbb{Z}}\) defined by

\[
\forall n \in \mathbb{Z} \quad w_n = r_n(t)s_n,
\]

is an homeomorphism such that:

\[
\forall k \in \mathbb{N} \quad |w|_k = |s|_k.
\]
As a consequence of this observation, of the expression of the continuity of \( \Lambda_T \) given by (3.8) and, of Parseval formula, we have that:

\[
(\exists k \in \mathbb{N}, \ c_k > 0) \quad (\forall \varphi \in \mathcal{S}(\mathbb{R})), \quad (\forall t \in [0,1])
\]

\[
\mathbb{E}\left[ \sum_{n=0}^{+\infty} r_n(t) a_n(T) a_n(\varphi) \right] \leq c_k \sup_{n \in \mathbb{N}} (1 + n)^k |a_n(\varphi)|.
\]

In the third step of the proof we will show that, the left-hand side of the inequality in (3.10) can be replaced by the expression:

\[
\mathbb{E}\left[ \sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2 \right]^{\frac{1}{2}}.
\]

In order to do as stated, we observe that, as the sequence \((a_n(\varphi))_{n \in \mathbb{N}}\) is rapidly decreasing and a. s. \((a_n(T))_{n \in \mathbb{N}}\) is a sequence of slow growth, we have almost surely:

\[
\sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2 < +\infty.
\]

Now by the standard inequality for Rademacher functions [25, p. 213], we have a. s. for some constant \(c\):

\[
(\sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2)^{\frac{1}{2}} \leq c \int_0^1 |\sum_{n=0}^{+\infty} r_n(t) a_n(T) a_n(\varphi)| dt.
\]

To conclude as desired, it is enough to apply Fubini theorem, to get, for \(k, c_k\) and \(\varphi\) as in (3.10):

\[
\mathbb{E}\left[ \sum_{n=0}^{+\infty} \sqrt{\alpha_n} |a_n(T)| |a_n(\varphi)| \right] \leq c_k \sup_{n \in \mathbb{N}} (1 + n)^k a_n(\varphi).
\]

In the fourth step, we remark that the left-hand side of the inequality in (3.11) can be replaced by

\[
\mathbb{E}\left[ \sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2 \right] = \sum_{n=0}^{+\infty} \alpha_n |a_n(T)|^2 \frac{|a_n(\varphi)|^2}{\alpha_n},
\]

where \((\alpha_n)_{n \in \mathbb{N}}\) is an arbitrary sequence of strictly positive, numbers such that \(\sum_{n=0}^{+\infty} \alpha_n = 1\). This statement results from the fact that, for almost every \(\omega \in \Omega\), the expression

\[
\sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2 = \sum_{n=0}^{+\infty} \alpha_n |a_n(T)|^2 \frac{|a_n(\varphi)|^2}{\alpha_n},
\]
can be considered as an integral over $\mathbb{N}$, of the function defined by:

$$\forall n \in \mathbb{N} \quad |a_n(T)|^2 \frac{|a_n(\varphi)|^2}{\alpha_n},$$

with respect to the measure over $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ that puts a mass $\alpha_n$ on each integer $n$. Applying Jensen inequality with the convex function $-\sqrt{x}$ on the interval $[0, +\infty[$, we have that a. s.:

$$\left(\sum_{n=0}^{+\infty} |a_n(T)|^2 |a_n(\varphi)|^2 \right)^{1/2} \geq \sum_{n=0}^{+\infty} \alpha_n |a_n(T)| \frac{|a_n(\varphi)|}{\sqrt{\alpha_n}}.$$

As a consequence, for $k, c_k$ and $\varphi$ as in (3.10):

$$\mathbb{E}\left[\sum_{n=0}^{+\infty} \sqrt{\alpha_n} |a_n(T)||a_n(\varphi)|\right] \leq c \sup_{n \in \mathbb{N}} (1 + n)^k |a_n(\varphi)|.$$

This expression shows that the sequence $(\mathbb{E}[|a_n(T)|\sqrt{\alpha_n}])_{n \in \mathbb{N}}$ is of slow growth at infinity. In order to conclude now, it is enough to consider for instance, the sequence $(\alpha_n)_{n \in \mathbb{N}}$ defined by:

$$\alpha_n = \begin{cases} \frac{1}{A} & n = 0 \\ \frac{1}{n^2 A} & n \neq 0 \end{cases}$$

where $A = 1 + \frac{x^2}{2}$. This sequence satisfies the hypothesis made in the fourth step and, it is clear that if, with this sequence, $(\mathbb{E}[|a_n(T)|\sqrt{\alpha_n}])_{n \in \mathbb{N}}$ is a sequence of slow growth at infinity then, $(\mathbb{E}[|a_n(T)|])_{n \in \mathbb{N}}$ is also of slow growth thus showing that, the random Schwartz distribution $T$ is in the class $C_m$. □

As a consequence of this theorem we can now formulate the result which gives a characterization of random Schwartz distributions having a mean or a first moment.

**Theorem 9.** Let $T$ be a measurable random Schwartz distribution. Then, $T$ has a first moment if and only if the sequence of its Hermite coefficients is in the class $C_m$.

**Proof.** That the last condition is sufficient was already shown in theorem 7. The condition is necessary as a consequence of the theorem 8. □

**Remark 1.** This last result can also be read as a characterization of the stochastic processes with a first moment which have as trajectories tempered distributions. In fact, let $X$ be a generalized stochastic process (see [11, p. 115]). This will mean for us and, according to the reference quoted that, $X = (X_\varphi)_{\varphi \in \mathcal{S}}$ is a family of random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for all $\omega \in \Omega$, the map from $\mathcal{S}$ into $\mathbb{C}$ given by $X_\varphi(\omega)$, is a tempered
distribution. We can then consider that $X$, has as trajectories tempered distributions. Obviously, such an object defines a random distribution in the sense we have been using and, moreover, the sequence $(a_m(X))_{m \in \mathbb{N}}$ is a sequence of random variables. As a consequence, theorem 9 can be applied giving the characterization mentioned.

4. TWO EXAMPLES

4.1. A brownian type process on $\mathbb{R}$. A gaussian white noise on $\mathbb{R}$, can naturally be presented using the concepts studied so far. Let $(\xi_m)_{m \in \mathbb{N}}$ be a normal sequence. That is, a sequence of independent, identically distributed, centered, gaussian, random variables, with variance equal to one. As we have:

$$\forall m \in \mathbb{N} \quad \mathbb{E}[|\xi_m|] = \sqrt{\frac{2}{\pi}},$$

the sequence is in the class $C_m$ and, being so:

(4.1) $$W = \sum_{m=0}^{+\infty} \xi_m \mathcal{H}_m,$$

defines in the usual way a random Schwartz distribution. Let us observe that, as a consequence of theorem 7:

$$\mathbb{E}[W] = \sum_{m=0}^{+\infty} \mathbb{E}[\xi_m] \mathcal{H}_m = 0.$$ 

The next easy proposition shows that, $W$ can be seen as a white noise on $\mathbb{R}$.

**Theorem 10.** $W$ can be extended as an isometry between $L^2(\mathbb{R})$ and the gaussian space $\mathcal{H}$ which is the closure in $L^2(\Omega)$ of the vector space generated by $(\xi_m)_{m \in \mathbb{N}}$.

**Proof.** Let us observe first that, for $\varphi \in \mathcal{S}$:

$$\langle W, \varphi \rangle = \sum_{m=0}^{+\infty} a_m(\varphi) \xi_m,$$

is a zero mean gaussian random variable. Using the orthonormality properties of the Hermite functions, the variance of this random variable is given by:

(4.2) $$\mathbb{E}[|\langle W, \varphi \rangle|^2] = \sum_{m=0}^{+\infty} |a_m(\varphi)|^2 = ||\varphi||^2.$$
Let now be \( f \in L^2(\mathbb{R}) \); the Hermite functions being a complete orthonormal system in \( L^2(\mathbb{R}) \) we have that:

\[
\|f\|^2 = \left( \sum_{m=0}^{+\infty} |a_m(f)|^2 \right)^{\frac{1}{2}} < +\infty.
\]

As a result, we can consider the extension of \( W \) to the whole \( L^2(\mathbb{R}) \) by taking for such an \( f \):

\[
W(f) = \sum_{m=0}^{+\infty} a_m(f) \xi_m.
\]

the series being almost surely convergent by an application of a known theorem (see [20, p. 58]). It is straightforward to verify that, \( W \) is an isometry (with the same calculation as the one which was done in formula 4.2). The following formulas:

\[
\forall m \in \mathbb{N}, W(\mathcal{H}_m) = \xi_m,
\]

show that \( W(L^2(\mathbb{R})) = \mathcal{H} \).

In order to define a brownian type process in \( \mathbb{R} \), we use, in a standard way, the gaussian white noise just constructed.

**Theorem 11.** There is an (unique in law) brownian type process \( (B_t)_{t \in \mathbb{R}} \) having a version with continuous trajectories such that:

\[
\forall t \in \mathbb{R}, B_t = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \, \xi_m \text{ a. s. on } \Omega.
\]

where \((\xi_m)_{m \in \mathbb{N}}\) is a given normal sequence.

**Remark 2.** For each \( t \in \mathbb{R} \), \( B_t \) is well defined as:

\[
(4.3) \quad B_t = \frac{1}{\sqrt{2}} W(\mathbb{I}_{[-|t|, +|t|]}).
\]

where \( W \), is the gaussian white noise of the previous section and, the indicator function of the interval \([-|t|, +|t|]\) obviously belongs to \( L^2(\mathbb{R}) \).

**Lemma 2.** For each \( t \), \( B_t \) is a centered gaussian random variable and, given \( s, t \in \mathbb{R} \), the covariance between \( B_s \) and \( B_t \) verifies:

\[
(4.4) \quad \mathbb{E}[B_t, B_s] = \min(|s|, |t|).
\]
Proof. [of the lemma] The fact that $B_t$ is a centered gaussian variable stems from the definition of the gaussian white noise $\mathcal{W}$. In order to verify formula 4.4, let us observe that, for every $t \in \mathbb{R}$:

$$
\frac{1}{\sqrt{2}} \mathbb{I}_{[-|t|, +|t|]} = \frac{1}{\sqrt{2}} \sum_{m = 0}^{+\infty} \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \mathcal{H}_m.
$$

As a similar representation holds replacing $t$ by $s \in \mathbb{R}$, we then have, as there is absolute convergence of the series involved:

$$
\frac{1}{2} (\mathbb{I}_{[-|t|, +|t|]} \times \mathbb{I}_{[-|s|, +|s|]}) = \frac{1}{2} \sum_{m, n \in \mathbb{N}} \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \left( \int_{-|s|}^{+|s|} \mathcal{H}_m(v) \, dv \right) \mathcal{H}_m \mathcal{H}_n.
$$

Now, integrating both sides of this equality in $\mathbb{R}$ gives:

$$\min(|s|, |t|) = \frac{1}{2} \sum_{m = 0}^{+\infty} \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \left( \int_{-|s|}^{+|s|} \mathcal{H}_m(v) \, dv \right),$$

as a by-product or the orthonormality relations among the Hermite functions. In order to conclude, it is enough to observe that:

$$
\mathbb{E}[B_t, B_s] = \frac{1}{2} \mathbb{E} \left[ \sum_{m = 0}^{+\infty} \left( \int_{-|s|}^{+|s|} \mathcal{H}_m(v) \, dv \right) \xi_m \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \xi_n \right]
$$

$$= \frac{1}{2} \sum_{m, n \in \mathbb{N}} \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \left( \int_{-|s|}^{+|s|} \mathcal{H}_m(v) \, dv \right) \mathbb{E}[\xi_m, \xi_n]
$$

$$= \min(|s|, |t|).$$

where the last equality results from the fact that the sequence $(\xi_m)_{m \in \mathbb{N}}$ is a normal sequence an so $\mathbb{E}[\xi_m, \xi_n] = \delta_{m,n}$. \hfill \square

Proof. [of the theorem] Let us notice that, the kernel $k(s, t) = \min(|s|, |t|)$ is a positive definite kernel as the following easy computation shows. Take $I$ a finite set, a family $(\alpha_i)_{i \in I}$ of complex numbers and, the counting measure $\mu_c$ over $(I, 2^I)$. Then, for any family $(s_i)_{i \in I}$ of real numbers and by Fubini theorem:

$$
\sum_{(i, j) \in I^2} k(s_i, s_j) \alpha_i \bar{\alpha}_j = \int_{I^2} \left( \int_{-\infty}^{+\infty} \mathbb{I}_{[0, |s_i|]}(u) \mathbb{I}_{[0, |s_j|]}(u) \right) \alpha_i \bar{\alpha}_j \, d\mu_c(i) \, d\mu_c(j) =
$$

$$= \int_{-\infty}^{+\infty} \left( \int_{I} \mathbb{I}_{[0, |s_i|]}(u) \alpha_i \, d\mu_c(i) \right) \left( \int_{I} \mathbb{I}_{[0, |s_j|]}(u) \bar{\alpha}_j \, d\mu_c(j) \right) \, du =
$$

$$= \int_{-\infty}^{+\infty} \left| \int_{I} \mathbb{I}_{[0, |s_i|]}(u) \alpha_i \, d\mu_c(i) \right|^2 \, du \geq 0.
$$

(4.5)
As a consequence, applying a well known result [16, p. 39], there is an unique (in law) gaussian process, with index set $\mathbb{R}$, having as a mean function the zero function and, as a covariance function the kernel $k(s,t)$.

**Remark 3.** $(B_t)_{t \in \mathbb{R}}$ could also be obtained by considering the usual brownian process in $(\tilde{B}_t)_{t \in \mathbb{R}^+}$ (see [14, p. 233]) and, posing as definition:

$$\forall t \in \mathbb{R} \quad B_t = \tilde{B}_{\lfloor t \rfloor} \quad \text{a. s. on } \Omega.$$ 

In fact, the following covariance computation for $s, t \in \mathbb{R}$:

$$\mathbb{E}[\tilde{B}_{\lfloor t \rfloor} \tilde{B}_{\lfloor s \rfloor}] = \min(|s|, |t|) = \mathbb{E}[B_t B_s],$$

shows that $(B_t)_{t \in \mathbb{R}}$ and $(\tilde{B}_{\lfloor t \rfloor})_{t \in \mathbb{R}}$ have the same law.

As a consequence of this remark $(B_t)_{t \in \mathbb{R}}$, can be thought as, the process in $\mathbb{R}$ obtained by pasting together an usual brownian process with its symmetrized version with respect to the $y$-axis. As a second consequence, let's point that $(B_t)_{t \in \mathbb{R}}$ admits a continuous version. This follows from the fact that $(\tilde{B}_t)_{t \in \mathbb{R}^+}$ admits a continuous version.

The process $(B_t)_{t \in \mathbb{R}}$ has a relation with $W$, similar with the one that the usual brownian process has with the gaussian white noise. That is the content of the following proposition.

**Theorem 12.** The stochastic process $B = (B_t)_{t \in \mathbb{R}}$, defines a random Schwartz distribution by taking for $\varphi \in \mathcal{S}$:

$$<B, \varphi> = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} \left( \int_{-\infty}^{+\infty} \varphi(t) \left( \int_{-|t|}^{+|t|} \mathcal{H}_m(u) \, du \right) \, dt \right) \xi_m,$$

and $B'$, the derivative of $B$ in the sens of distributions verifies:

$$B' = \frac{1}{\sqrt{2}} (W + W^\vee),$$

where the operator $\vee$ is the transposed operator of the operator acting on a $\varphi \in \mathcal{S}$ by:

$$\forall t \in \mathbb{R} \quad \varphi^\vee(t) = \varphi(-t).$$
Proof. By Fubini theorem, the integral on the right-hand side of definition 4.6 can be rewritten in the following form:

\[ \int_{-\infty}^{+\infty} \mathcal{H}_m(u) \left( \int_{-\infty}^{+\infty} \mathbb{I}_{[-|t|,+|t|]}(u) \varphi(t) \, dt \right) \, du = \]
\[ = \int_{-\infty}^{0} \mathcal{H}_m(u) \left( \int_{-\infty}^{0} \mathbb{I}_{[-\infty,|+t|]}(t) \varphi(t) \, dt + \int_{0}^{+\infty} \mathbb{I}_{[-u,\infty]}(t) \varphi(t) \, dt \right) \, du + \]
\[ + \int_{0}^{+\infty} \mathcal{H}_m(u) \left( \int_{-\infty}^{0} \mathbb{I}_{[-\infty,-|u|]}(t) \varphi(t) \, dt + \int_{0}^{+\infty} \mathbb{I}_{[u,\infty]}(t) \varphi(t) \, dt \right) \, du. \]

Now, let us take for instance, the first integral in the integrand part of the right-hand side of this last formula and, define:

\[ F_1(u) = \left( \int_{-\infty}^{0} \mathbb{I}_{[-\infty,|+u|]}(t) \varphi(t) \, dt \right) \mathbb{I}_{[-\infty,0]}(u). \]

Using the the hypothesis made on \( \varphi \), namely that \( \varphi \in \mathcal{S} \) and, as a result of an easy application of Cauchy-Schwarz inequality and Fubini theorem, we get \( F_1 \in L^2(\mathbb{R}) \). In fact, we have:

\[ |F_1(u)| \leq \left( \int_{-\infty}^{u} (1 + |t|)^{\frac{3}{2}} |\varphi(t)| \frac{dt}{(1 + |t|)^{\frac{3}{2}}} \right) \mathbb{I}_{[-\infty,0]}(u) \]
\[ \leq \left( \int_{-\infty}^{u} (1 + |t|)^{3} |\varphi(t)|^2 \, dt \right)^{\frac{1}{2}} \left( \int_{-\infty}^{u} \frac{dt}{(1 + |t|)^{3}} \right)^{\frac{1}{2}} \mathbb{I}_{[-\infty,0]}(u). \]

Now, considering the function \( G_1 \) defined by:

\[ (4.10) \quad G_1(u) = \left( \int_{-\infty}^{u} \frac{dt}{(1 + |t|)^{3}} \right)^{\frac{1}{2}} \mathbb{I}_{[-\infty,0]}(u), \]

we have by Fubini theorem that:

\[ \int_{\mathbb{R}} |G_1(u)|^{2} \, du = \int_{-\infty}^{0} \left( \int_{-\infty}^{u} \frac{dt}{(1 + |t|)^{3}} \right) \, du = \int_{-\infty}^{0} \frac{|t|}{(1 + |t|)^{2}} \, dt < +\infty. \]

We have just shown that \( G_1 \) and, as a consequence \( F_1 \), are \( L^2(\mathbb{R}) \) functions. Applying the same idea to the similar integral terms in the integrand part of the right-hand side of 4.9, we can conclude that in formula 4.6, the coefficient of \( \xi_m \) in the sum is a Hermite coefficient of an \( L^2 \) function an so, the series converges almost surely. Let us make this idea more precise by denoting \( F_2, F_3 \) and \( F_4 \), (respectively \( G_2, G_3 \) and \( G_4 \)) the functions just refered as the integral
terms, similar to $F_1$ in 4.9, (respectively to $G_1$ in 4.10). Then, denoting by $F_\varphi$ the function defined by:

$$F_\varphi(u) = \int_{-\infty}^{+\infty} \mathbb{I}_{[-|u|,|u|]}(u) \varphi(t) \, dt ,$$

we have that, $F_\varphi = \sum_{i=1}^4 F_i$ with $F_i \in L^2(\mathbb{R})$ for $i \in \{1, \ldots, 4\}$. Furthermore, by using similar estimates for $F_i$ and $G_i$ with $i = 2, \ldots, 4$, as the estimates proved for $F_1$ and $G_1$, we have that, for some constant $C$:

$$(4.11) \quad ||F_\varphi||_2 \leq C \sup_{t \in \mathbb{R}} \left((1 + |t|)^3 |\varphi(t)|\right) \left(\sum_{i=1}^4 ||G_i||_2 \right) < +\infty .$$

Now, just observe that, by the first line in formula 4.9,

$$< \mathcal{B}, \varphi > = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} < F_\varphi, \mathcal{H}_m > \xi_m = \frac{1}{\sqrt{2}} W(F_\varphi) .$$

This formula shows that $< \mathcal{B}, \varphi >$ is defined by a series which converges almost surely, as a consequence of having $F_\varphi \in L^2(\mathbb{R})$. Furthermore, inequality 4.11 shows that the map from $S$ into $L^2(\mathbb{R})$ which associates to the test function $\varphi \in S$, the function $F_\varphi$ is continuous. As $W$ is continuous from $L^2(\mathbb{R})$ into $S$, then $\mathcal{B}$ is continuous from $S$ into $S$ and so, $\mathcal{B}$ is a random Schwartz distribution. In order to compute the derivative of $\mathcal{B}$ it is enough to recall that the derivative of $\mathcal{B}$ is given, as usually, for $\varphi \in S$ by:

$$< \mathcal{B}', \varphi > = - < \mathcal{B}, \varphi' > .$$

Using the definition of $\mathcal{B}$ given by 4.6 and, after some integrations by parts on the terms obtained by considering the case $t \geq 0$ and the case $t \leq 0$, we get that:

$$\mathcal{B}' = \frac{1}{\sqrt{2}} \sum_{m=0}^{+\infty} (\mathcal{H}_m(t) + \mathcal{H}_m(-t)) \xi_m \text{ a. s. on } \Omega .$$

This expression shows, after some change of variables to deal with the series having as coefficients $(\mathcal{H}_m(-t))_{m \in \mathbb{N}}$ that:

$$\mathcal{B}' = \frac{1}{\sqrt{2}} (W + W^\gamma) \text{ a. s. on } \Omega ,$$

which is the result announced in 4.7. □

Remark 4. The usual Brownian process on $\mathbb{R}_+$, let it be $\tilde{\mathcal{B}} = (\tilde{B}_t)_{t \in [0, +\infty[}$ could have been studied in a similar fashion, using the gaussian white noise $W$ by taking as a definition:

$$\tilde{B}_t = W(\mathbb{I}_{[0,t]}).$$
Instead of formula 4.8 in theorem 12, we would then have as usually:

$$\frac{d}{dt} \tilde{\mathbf{B}} = \mathcal{W} \text{ a. s. on } \Omega,$$

With this definition, the process $\tilde{\mathbf{B}}$ would be a bona-fide Brownian process satisfying all the properties of Brownian processes (see [20, p. 220]. The process $\mathbf{B}$ just studied, does not satisfy the independence and distribution conditions on the increments.

4.2. The brownian distributions. A natural generalization of the construction presented in the last subsection, consists on taking as a starting point an isometry between $L^2(\mathbb{R}, \mu)$, where $\mu$ is a tempered measure on $\mathbb{R}$ with the Borel $\sigma$-algebra and, a gaussian space $\mathcal{G}$ of $L^2(\Omega)$. A theorem of J.-P. Kahane, which we will state and prove next (see theorem 14) will show that this generalization gives raise to some random Schwartz distributions of the type we have been studying. To begin with, let us make some remarks on the properties of tempered measures needed in the sequel.

**Definition 4.** A positive measure $\mu$ on $\mathbb{R}$, is a tempered measure if and only if:

$$\exists l \in \mathbb{N} \quad \int_{\mathbb{R}} \frac{d\mu}{(1 + |t|^2)^l} < +\infty.$$  

It is obvious that $\mu$ integrates the function $1/(1 + |t|^2)^{l+\epsilon}$, with $\epsilon > 0$, whenever $\mu$ integrates $1/(1 + |t|^2)^l$ and so, if we define:

$$l_\mu = \min\{ l \in \mathbb{N} : \int_{\mathbb{R}} \frac{d\mu}{(1 + |t|^2)^l} < +\infty \},$$

then, $l_\mu$ is a well defined integer for every tempered measure. In some sense, this integer quantifies the growth of the measure $\mu$ at infinity.

Let us state and prove now a classical structure result (see [24, II. p. 255]) in a form which will be usefull for our purposes.

**Theorem 13.** Let $\mu$ be a positive tempered measure, over $\mathbb{R}$. Let $l_\mu \in \mathbb{N}$ be the integer given by formula 4.12. There exists then $f \in L^2(\mathbb{R})$ such that, in the sense of tempered distributions, we have:

$$\mu = ((1 + |t|^2)^{l_\mu+1} f)'$$

This means, by definition, that:

$$\forall \varphi \in S \quad \langle \mu, \varphi \rangle = \int_{\mathbb{R}} \varphi \, d\mu = \int_{\mathbb{R}} (1 + |t|^2)^{l_\mu+1} f(t) \varphi'(t) \, dt.$$
Proof. Let us show first that, $\mu$ defines in the first equality of the formula 4.13 a tempered distribution. Indeed for $\varphi \in \mathcal{S}$:

$$
| < \mu, \varphi > | \leq \int_{\mathbb{R}} (1 + |t|^2)^{\nu - 1} |\varphi(t)| \frac{d\mu(t)}{(1 + |t|^2)^{\nu}} \\
\leq \sup_{t \in \mathbb{R}} \left( (1 + |t|^2)^{\nu - 1} |\varphi(t)| \right) \int_{\mathbb{R}} \frac{d\mu(t)}{(1 + |t|^2)^{\nu}}.
$$

This inequality shows that $\mu$ defines a tempered distribution. Now, it is possible to prove (see [24, II, p. 253]) that, for every $\varphi \in \mathcal{S}$:

$$
\sup_{t \in \mathbb{R}} \left( (1 + |t|^2)^{\nu - 1} |\varphi(t)| \right) \leq \| (1 + |t|^2)^{\nu - 1} \varphi'(t) \|_{1} \leq \| (1 + |t|^2)^{\nu} \varphi'(t) \|_{2}.
$$

And, being so, we have:

$$
| < \mu, \varphi > | \leq \left( \int_{\mathbb{R}} \frac{d\mu(t)}{(1 + |t|^2)^{\nu}} \right) \| (1 + |t|^2)^{\nu - 1} \varphi'(t) \|_{2}.
$$

This shows that $\mu$ is an element of the dual of $X$, the space of $C^\infty(\mathbb{R})$ functions with compact support, endowed with the topology associated to the prehilbertian norm given by:

$$
\| \varphi \| = \| (1 + |t|^2)^{\nu + 1} \varphi'(t) \|_{2}.
$$

As the map which associates to $\varphi \in X$ the function $(1 + |t|^2)^{\nu + 1} \varphi'(t)$ is an isometric embedding of $X$ into $L^2(\mathbb{R})$, its transpose is an surjective mapping from $L^2(\mathbb{R})$ onto $X'$ (see again [24, II, p. 249]). This shows that there exists $f \in L^2(\mathbb{R})$ satisfying formula 4.13 stated in the theorem, for every $\varphi$ in the test function space $\mathcal{D}$. This space $\mathcal{D}$ is the space of $C^\infty(\mathbb{R})$ functions having compact support, with the topology of uniform convergence on compact sets, not only of the functions but also of the derivatives of all orders of the functions. As the space $\mathcal{D}$ is dense in $\mathcal{S}$ (see [24, II, p. 7]), some usual integration arguments allow us to conclude. In fact, let $\varphi \in \mathcal{S}$ and $(\psi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}$ having $\varphi$ as a limit in $\mathcal{S}$. Then, as we have:

$$
\lim_{n \to +\infty} (1 + |t|^2)^{\nu} \psi_n = (1 + |t|^2)^{\nu} \varphi,
$$

the convergence being uniform in $\mathbb{R}$ and, $\int (1 + |t|^2)^{-\nu} d\mu < +\infty$, we can conclude that,

(4.14) 
$$
< \mu, \varphi > = \lim_{n \to +\infty} \int_{\mathbb{R}} \psi_n d\mu = \lim_{n \to +\infty} < \mu, \psi_n >.
$$
By Cauchy-Schwarz inequality it is also true that:

\[ \left| \int_{\mathbb{R}} (1 + |t|^2)^{l_{\mu}+1} f(t) \left( \psi_n(t) - \varphi(t) \right) dt \right| \leq \| f \|_2 \left( \int_{\mathbb{R}} \frac{dt}{1 + |t|^2} \right)^{\frac{1}{2}} \sup_{t \in \mathbb{R}} \left( (1 + |t|^2)^{l_{\mu}+\frac{3}{2}} |\psi_n(t) - \varphi(t)| \right). \]

This inequality shows that:

\[ \lim_{n \to +\infty} \int_{\mathbb{R}} (1 + |t|^2)^{l_{\mu}+1} f(t) \psi_n(t) dt = \int_{\mathbb{R}} (1 + |t|^2)^{l_{\mu}+1} f(t) \varphi(t) dt. \]

As a consequence of formulas 4.14 and 4.15 we have that, formula 4.13 is verified also for the test functions \( \varphi \) in \( \mathcal{S} \). \( \square \)

In order to prepare a lean redaction of the proof of the theorem on brownian distributions, we will apply this result to obtain estimates of the image by \( \mu \) of the generating function of the squares of Hermite functions.

**Lemma 3.** For \( 0 \leq t < 1 \) we have for some constant \( c = c(\mu) \):

\[ |< \mu, e^{-2\pi x^2(\frac{1}{1+t})} >| \leq c \left( \frac{1-t}{1+t} \right)^{(l_{\mu}+\frac{3}{2})}. \]

**Proof.** Using formula 4.13 we have that:

\[ |< \mu, e^{-2\pi x^2(\frac{1}{1+t})} >| \leq 4\pi \left( \frac{1-t}{1+t} \right) \int_{\mathbb{R}} (1 + |x|^2)^{l_{\mu}+1} |x||f(x)||e^{-2\pi x^2(\frac{1}{1+t})} dx. \]

We decompose the integral of the right-hand term of this inequality into the sum of the integral over \([-1, +1]\)and the integral over the complement of this set. The first integral to consider gives by Cauchy-Schwarz inequality:

\[ \int_{|x| \leq 1} e^{-ax^2} |f(x)| 2^{l_{\mu}+1} dx \leq 2^{l_{\mu}+1} \| f \|_2 \left( \int_{|x| \leq 1} e^{-ax^2} dx \right)^{\frac{1}{2}}, \]

denoting by:

\[ a = 2\pi \left( \frac{1-t}{1+t} \right) > 0. \]

Recall now that, for every \( l \in \mathbb{N} \):

\[ \int_{\mathbb{R}} e^{-x^2} x^{2l} dx = \sqrt{\pi} \frac{(2l)!}{4^l l!}. \]

And, this implies after a change of variables, that:

\[ \left( \int_{|x| \leq 1} e^{-ax^2} dx \right)^{\frac{1}{2}} \leq 2^{-\frac{1}{4}} \left( \frac{1-t}{1+t} \right)^{-\frac{1}{4}}, \]
giving finally the estimate for the first integral:

$$4\pi \left( \frac{1 - t}{1 + t} \right) \int_{|x| \leq 1} (1 + |x|^2)^{1/2} |x| |f(x)| e^{-2\pi x^2} \frac{1 - t}{1 + t} \, dx \leq \pi 2^{l_\mu + \frac{14}{3}} \|f\|_2 \left( \frac{1 - t}{1 + t} \right)^{\frac{3}{4}}.$$ 

For the integral over $[-1, +1]^c$ and, with the same notations as before, we have:

$$\int_{|x| > 1} |x| e^{-a x^2} (1 + |x|^2)^{l_\mu + 1} |f(x)| \, dx \leq \left( \int_{|x| > 1} |x|^{2l_\mu + 3} e^{-a x^2} |f(x)| \, dx \right)^{\frac{1}{2}} \leq \left( \int_{|x| > 1} x^{4l_\mu + 6} e^{-2a x^2} \, dx \right)^{\frac{1}{2}} \|f\|_2.$$ \hspace{1cm} (4.18)

Using again formula 4.17, with a change of variables, we get that:

$$4\pi \left( \frac{1 - t}{1 + t} \right) \left( \int_{|x| > 1} x^{4l_\mu + 6} e^{-2a x^2} \, dx \right)^{\frac{1}{2}} \leq \frac{1}{2^{3l_\mu + \frac{1}{2}} \pi^{l_\mu + \frac{1}{2}}} \sqrt{\frac{(4l_\mu + 6)!}{(2l_\mu + 3)!}} \left( \frac{1 - t}{1 + t} \right)^{-\left( l_\mu + \frac{3}{4} \right)} \|f\|_2.$$ 

In order conclude as stated in the lemma, just consider

$$c = 2 \|f\|_2 \max \left( \pi 2^{l_\mu + \frac{14}{3}}, \frac{1}{2^{3l_\mu + \frac{1}{2}} \pi^{l_\mu + \frac{1}{2}}} \sqrt{\frac{(4l_\mu + 6)!}{(2l_\mu + 3)!}} \left( \frac{1 - t}{1 + t} \right)^{-\left( l_\mu + \frac{3}{4} \right)} \right),$$

and observe that:

$$\left( \frac{1 - t}{1 + t} \right)^{\frac{3}{4}} \leq \left( \frac{1 - t}{1 + t} \right)^{-\left( l_\mu + \frac{3}{4} \right)},$$

for $0 \leq t < 1$ and $l_\mu \in \mathbb{N}$. □

We are now ready to state and prove a theorem of J.-P. Kahane [13, p. 121] which, in the context in which we have been working, allow us, to construct important examples of random Schwartz distributions. The proof presented, follows the main idea of the original proof but, relies in the work presented above in order to obtain accurate estimates of the order of growth of the random Schwartz distribution built.

**Definition 5.** Given a tempered measure $\mu$, a $\mu$-brownian distribution is an isometry between $L^2(\mathbb{R}, \mu)$ and $\mathcal{H}$ a gaussian closed subspace of $L^2(\Omega)$ where $(\Omega, \mathcal{A}, \mathbb{P})$, is supposed to be a complete probability space with no atoms.
This definition makes good sense, since as $\mu$ is a tempered measure on $\mathbb{R}$, the Hilbert space $L^2(\mathbb{R}, \mu)$ is separable. Moreover, as a consequence of the hypotheses made on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, there exist gaussian, closed, separable and non trivial, subspaces of $L^2(\Omega)$. By choosing complete orthonormal sequences in both $L^2(\mathbb{R}, \mu)$ and $\mathcal{H}$, let them be respectively $(f_n)_{n \in \mathbb{N}}$ and $(\zeta_n)_{n \in \mathbb{N}}$ and, associating naturally $f_n$ to $\zeta_n$ for each $n \in \mathbb{N}$, there exists an isometry $Z_\mu$ between $L^2(\mathbb{R}, \mu)$ and $\mathcal{H}$ given by:

$$(4.19) \quad Z_\mu(f) = \sum_{m=0}^{+\infty} < f, f_n > \zeta_n,$$

where $(< f, f_n >)_{n \in \mathbb{N}}$ is the family of the Fourier coefficients of $f$ with respect to $(f_n)_{n \in \mathbb{N}}$. This family verifies:

$$f = \sum_{m=0}^{+\infty} < f, f_n > f_n,$$

with equality in $L^2(\mathbb{R}, \mu)$ sense, the series 4.19 being convergent in $L^2(\Omega)$. Observe incidentally that as:

$$\|f\|_2^2 = \sum_{m=0}^{+\infty} | < f, f_n > |^2 < +\infty,$$

the series 4.19 converges almost surely.

**Theorem 14 (On brownian distributions).** Let $\mu$ be a tempered positive measure and $Z_\mu$ a $\mu$-brownian distribution. There exists then, a random Schwartz distribution $Z^*$ such that $Z_\mu = Z^*$ almost surely.

**Proof.** As we have that:

$$\forall m \in \mathbb{N} \quad \mathcal{H}_m \in \mathcal{S} \subset L^2(\mathbb{R}, \mu),$$

it will be enough to show that the sequence $(Z_\mu(\mathcal{H}_m))_{m \in \mathbb{N}}$ is a sequence of random variables in the class $\mathcal{C}_m$. As we already know, we will then have that, $Z^*$ defined by:

$$Z^* = \sum_{m=0}^{+\infty} Z_\mu(\mathcal{H}_m) \mathcal{H}_m,$$

is a well defined random Schwartz distribution. By the duality formula 2.4 we will have, using the fact that $Z_\mu$ is an isometry and, being so, it is a linear and continuous map that, for $\varphi \in \mathcal{S}$:

$$< Z^*, \varphi > = \sum_{m=0}^{+\infty} Z_\mu(\mathcal{H}_m) < \varphi, \mathcal{H}_m > = Z_\mu(\sum_{m=0}^{+\infty} < \varphi, \mathcal{H}_m > \mathcal{H}_m) = Z_\mu(\varphi).$$
In order to prove the statement made on \((Z_\mu(\mathcal{H}_m))_{n \in \mathbb{N}}\) we recall formula 2.5 which gives the generating function for the squares of Hermite functions. For \(|t| < 1\):

\[
\sum_{m=0}^{+\infty} t^m \mathcal{H}_m^2(x) = \left(\frac{2}{1 - t^2}\right)^{\frac{1}{2}} e^{-2\pi x^2 \frac{1 - t}{1 + t}}.
\]

Now, by the Lebesgue monotone convergence theorem, integrating both sides of this equality with respect to \(\mu\) and, using the fact that \(Z_\mu\) is an isometry, we have that, for \(0 < t < 1\):

\[
\sum_{m=0}^{+\infty} t^m \|Z_\mu(\mathcal{H}_m)\|_2^2 = \left(\frac{2}{1 - t^2}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-2\pi x^2 \frac{1 - t}{1 + t}} d\mu(x).
\]

As a consequence of the estimates obtained in formula 4.16, we get for \(p \geq 0\) and \(0 < t < 1\):

\[
(4.21) \quad t^p \|Z_\mu(\mathcal{H}_p)\|_2^2 \leq \sum_{m=0}^{+\infty} t^m \|Z_\mu(\mathcal{H}_m)\|_2^2 \leq c \left(\frac{2}{1 - t^2}\right)^{\frac{1}{2}} \left(\frac{1 - t}{1 + t}\right)^{-(\mu_0 + \frac{1}{2})}.
\]

Consider now that, for \(p \geq 1\) we have: \(t = 1 - 1/p\). We will have then, for some constant \(C\) and, after some computations on inequality 4.21, that:

\[
(4.22) \quad \forall p \geq 1 \quad \|Z_\mu(\mathcal{H}_p)\|_2^2 \leq C (2p - 1)^{\mu_0 + \frac{5}{4}}.
\]

In order to conclude, let us observe that the random variable

\[
\frac{Z_\mu(\mathcal{H}_p)}{\|\mathcal{H}_p\|},
\]

is a standard normal random variable; being so we have:

\[
\mathbb{E}\left[\left|Z_\mu(\mathcal{H}_p)\right| \right] = \frac{2}{\sqrt{2\pi}}.
\]

This, together with inequality 4.22, shows that:

\[
\exists c > 0 \quad \forall p \geq 1 \quad \mathbb{E}\left[\left|Z_\mu(\mathcal{H}_p)\right| \right] \leq c (2p - 1)^{\mu_0 + \frac{5}{4}},
\]

which tells us that, the sequence \((Z_\mu(\mathcal{H}_p))_{p \in \mathbb{N}}\) is in the class \(C_\mu\), fully justifying the statement of the theorem. \(\square\)
5. SOME REMARKS ON THE BROWNIAN DISTRIBUTION EXAMPLE

In this last section we are going to present some comments and results tied to the following problem.

**Problem 1.** Let $Z$ be a random Schwartz distribution of the type we have been studying. Under what conditions on $Z$ is there a tempered positive measure $\mu$ such that for the associated brownian distribution $Z_\mu$ we have $Z = Z_\mu$ almost surely?

As a consequence of the results presented so far, $(< Z, \mathcal{H}_m >)_{m \in \mathbb{N}}$ has to be a sequence of centered gaussian variables such that, the sequence of their variances $(\mathbb{E}[|< Z, \mathcal{H}_m >|^2])_{m \in \mathbb{N}}$, is a sequence of slow growth. Furthermore, as a consequence of formula 4.20, in the proof of Kahane’s theorem, we know that the following Hamburger type moment problem:

$$M_m = \frac{1}{m!} \int e^{-2\pi x^2 \frac{1+i}{1-i}} d\mu(x)$$

(5.1)

where $m \in \mathbb{N}$ and $M_m$ stands for $\mathbb{E}[|< Z, \mathcal{H}_m >|^2]$, must have as a solution a tempered measure. Let us try to pursue this line of reasoning a little further. Denoting by $G(t)$, $f(t)$ and $g_\mu(t)$ respectively

$$G(t) = \frac{1-t}{1+t}, \quad f(t) = \frac{2}{1-t^2} \quad \text{and} \quad g_\mu(t) = \int e^{-2\pi x^2 \frac{1+i}{1-i}} d\mu(x),$$

we have that, the moment problem formulated in equations 5.1 has a solution if for every $p \in \mathbb{N}$ the lower triangular linear system given by equations:

$$m = 0, \ldots, p \quad M_m = \sum_{k=0}^{m} \frac{1}{k! (m-k)!} f^{(m-k)}(0) g_\mu^{(p)}(0),$$

(5.2)

has a solution which we will denote by

$$(g_\mu(0), g_\mu'(0), \ldots, g_\mu^{(p)}(0)).$$

Let us observe that as a result of a straightforward calculation he have that for $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$:

$$f^{(m-k)}(0) = \begin{cases} 0 & \text{if } m-k \text{ is odd} \\ (1 \times 3 \times 5 \times \cdots \times (m-k-1))^2 \sqrt{2} & \text{if } m-k \text{ is even} \end{cases}$$

As a consequence, given a sequence $(M_n)_{n \in \mathbb{N}}$, the solution of the linear system given by equations 5.2, always exist and, can be computed by induction. Let us detail now the computations for $(g_\mu^{(m)}(0))_{m \in \mathbb{N}}$. It easy to see that, all the derivatives of $g_\mu$ taken at the point zero exist and, using the formula of Faa
di Bruno ([8, p. 261], [21, p. 248] or [2, p. I47]) which gives the derivative of order \( m \) of the composition of two maps, that:

\[
\frac{g^{(m)}_{\mu}(0)}{m! n_1! n_2! \ldots n_q!} \int_{\mathbb{R}} x^{2p} e^{-2\pi x^2} d\mu(x).
\]

In formula 5.3, the sum is over all the integer sequences \((n_i)_{1 \leq i \leq q}\) such that \(\sum_{i=1}^{q} in_i = m\) and with \(\sum_{i=1}^{q} n_i = p\). This representation shows that, \(g^{n}_{\mu}(0)\) is a linear combination of moments of even order of \(d\nu = e^{-2\pi x^2} d\mu\). And so, if a tempered measure \(\mu\), solution of problem 1 exists, then, the even moments of \(d\nu\) must satisfy equations 5.3. Let us observe that, for every \(p \in \mathbb{N}\) the lower triangular linear system given by equations 5.3 for \(m = 0, \ldots, p\), always has a solution. So, problem 1 stated in the introduction of this section can be rephrased in the following manner.

**Problem 2.** Given a sequence \((M_m)_{m \in \mathbb{N}}\) of slow growth does there exist a tempered measure \(\mu\) such that the even moments of \(d\nu = e^{-2\pi x^2} d\mu\) satisfy equations 5.3, in which the sequence \((g^{(m)}_{\mu}(0))_{m \in \mathbb{N}}\) is in turn a solution of equations 5.2?

At the sight of this statement of the problem, a natural question to ask is: what can go wrong since as we have already shown, a solution for the linear systems involved always exist? At the moment, the answer we can give has three main components:

1. The sequence obtained may not be a solution of the moment problem if it has a growth which is non compatible with the condition of \(\mu\) being a tempered measure. This growth is precised by proposition 15.

2. The Hamburger type moment problem in itself, that is the problem of the existence and uniqueness of a measure having given moments over \(\mathbb{R}\) may not be well posed.

3. The sequence obtained as solution of the linear systems may not be a sequence of even moments of a positive measure \(d\nu\) if for instance, it contains some negative terms.

In order to deal with the first component of the answer above, we may formulate a characterization of tempered measures which will provide a usefull testing criterium for measures which are solutions of problem 2.

**Theorem 15.** Let \(\mu\) be a positive measure on \(\mathbb{R}\). Then, given \(l \in \mathbb{N}\), a necessary and sufficient condition for having,

\[
\int_{\mathbb{R}} \frac{d\mu(x)}{(1 + |x|^2)^l} < +\infty
\]
is that:

\begin{equation}
\mu(\{ |x| < 1 \}) < +\infty
\end{equation}

and

\begin{equation}
(\exists (a_k)_{k \in \mathbb{N}} \in \mathbb{R}^N_+ \sum_{k=0}^{+\infty} (2\pi)^k a_k < +\infty) \quad (\forall k \in \mathbb{N})
\end{equation}

\begin{equation}
\int_{|x| \geq 1} x^{2k-2l} e^{-2\pi x^2} \, d\mu(x) \leq k! a_k .
\end{equation}

**Proof.** Suppose that 5.6 is verified. Then, by Lebesgue’s monotone convergence theorem:

\[
\int_{|x| \geq 1} \frac{d\mu}{(1 + |x|^2)^l} = \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x| \geq 1} x^{2k} e^{-2\pi x^2} \frac{d\mu}{(1 + |x|^2)^l} \\
\leq \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x| \geq 1} x^{2k-2l} e^{-2\pi x^2} d\mu \\
\leq \sum_{k=0}^{+\infty} (2\pi)^k a_k < +\infty ,
\]

as a consequence of having, for \(|x| \geq 1\), that:

\[
\frac{1}{(1 + |x|^2)^l} \leq \frac{1}{|x|^{2l}} .
\]

Now, observe that:

\begin{equation}
2^{-l} \mu(\{|x| < 1\}) = \int_{|x| < 1} \frac{d\mu}{2^l} \leq \int_{|x| < 1} \frac{d\mu}{(1 + |x|^2)^l} \leq \mu(\{|x| < 1\}) ,
\end{equation}

and obviously we can conclude that 5.5 and 5.6 do imply 5.4. Suppose now that 5.4 is verified. Then, certainly, 5.5 is verified and, the same decomposition as above yields:

\[
+\infty > \int_{|x| \geq 1} \frac{d\mu}{(1 + |x|^2)^l} = \sum_{k=0}^{+\infty} \frac{(2\pi)^k}{k!} \int_{|x| \geq 1} x^{2k} e^{-2\pi x^2} \frac{d\mu}{(1 + |x|^2)^l} .
\]

Denoting:

\[
b_k = \frac{1}{k!} \int_{|x| \geq 1} x^{2k} e^{-2\pi x^2} \frac{d\mu}{(1 + |x|^2)^l} ,
\]

it is true that:

\[
\sum_{k=0}^{+\infty} (2\pi)^k b_k < +\infty .
\]
In order to conclude, just observe that:

\[ 2^l \, k! \, b_k \geq \int_{|x| \geq 1} x^{2k-2l} e^{-2\pi x^2} \frac{d\mu}{x^{2l}} , \]

and so, that 5.6 is verified with \( a_k = 2^l \, b_k \). \( \square \)

To deal with the second component of our answer, we recall a result of Carleman (see [15, p. 289a] or [17, p. 292a]).

**Theorem 16 (Carleman).** The Hamburger moment problem of asserting the existence of a measure \( \nu \), verifying for a given sequence \( (N_m)_{m \in \mathbb{N}} \), the equations given by:

\[ N_m = \int_{\mathbb{R}} x^m \, d\nu(x) , \]

has an unique solution if:

\[
\sum_{m=0}^{+\infty} \frac{1}{2^m \sqrt{N_{2m}}} = +\infty .
\]

(5.8)

We observe next that there is a connection between the Carleman's condition 5.8 and the condition 5.6 of theorem 15.

**Lemma 4.** Let \( \mu \) be a tempered measure and \( l \in \mathbb{N} \) be an integer for which conditions 5.5 and 5.6 are satisfied. Then, denoting for \( n \in \mathbb{N} \),

\[ \mu_{2n} = \int_{\mathbb{R}} x^{2n} e^{-2\pi x^2} \, d\mu(x) , \]

we have:

\[
\sum_{n=0}^{+\infty} \frac{1}{2^m \sqrt{\mu_{2m}}} = +\infty .
\]

Proof. As a consequence of condition 5.6 we have:

\[
\int_{|x| \geq 1} x^{2(n+l)-2l} e^{-2\pi x^2} \, d\mu(x) = \int_{|x| \geq 1} x^{2n} e^{-2\pi x^2} \, d\mu(x) \leq a_{n+l} (n+l)! .
\]

This estimate together with condition 5.5 implies that:

\[
\frac{1}{\mu_{2n}} \geq \frac{1}{\mu(|x| \geq 1)} + a_{n+l} (n+l)! .
\]

Finally, using for instance Stirling's formula it is straightforward to prove that:

\[
\frac{1}{2^m \sqrt{\mu_{2m}}} \simeq \frac{1}{\sqrt{n}} ,
\]

and the conclusion of the lemma is then verified. \( \square \)
As a consequence of theorem 16 and, using a similar idea as the one presented in lemma 4, it is possible to establish a criterium that gives a partial conclusion to the first and second components of the answer given to the question stated after problem 2.

**Theorem 17.** Let \((N_m)_{m \in \mathbb{N}}\) be a sequence of nonnegative numbers. Then the following propositions are equivalent.

1. For a certain \(l \in \mathbb{N}\), there exists \((a_k)_{k \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}_+\), a sequence of nonnegative numbers such that \(\sum_{k=0}^{+\infty} (2\pi)^k a_k < +\infty\), which verifies:

\[
(\forall k \in \mathbb{N}) \quad N_{2k} \leq a_{k+l} (k+l)! .
\]

2. There exists an unique tempered measure \(\mu\) such that:

\[
(\forall k \in \mathbb{N}) \quad N_{2k} = \int_{\mathbb{R}} x^{2k} e^{-2\pi x^2} \, d\mu(x) .
\]

**Proof.** With the hypotheses made in proposition 1 we have that, for \(\epsilon > 0\) given:

\[
\exists k_0 \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad k \geq k_0 \quad \Rightarrow \quad a_k \leq \frac{\epsilon}{(2\pi)^k} .
\]

As a consequence, we get for \(k \geq k_0\):

\[
\frac{1}{2^{k} \sqrt{N_{2k}}} \geq \frac{1}{2^{k} \sqrt{a_{k+l} (k+l)!}} \geq \sqrt{\frac{(2\pi)^{k+1}}{\epsilon (k+l)!}} \approx \frac{1}{\sqrt{k}} ,
\]

and so:

\[
\sum_{m=0}^{+\infty} \frac{1}{2^{k} \sqrt{N_{2k}}} = +\infty .
\]

By Carleman's theorem, this relation is now sufficient to ensure the existence of an unique measure \(\nu\), such that:

\[
N_{2k} = \int_{\mathbb{R}} x^{2k} \, d\nu(x) .
\]

Considering now the measure \(\mu\) defined by:

\[
d\mu(x) = e^{2\pi x^2} \, d\nu(x) ,
\]

it is clear that formula 5.10 is verified. It remains to be proved that \(\mu\) is a tempered measure. For that purpose, just observe that:

\[
(\forall n \geq l) \quad \int_{|x| \geq 1} x^{2n-2l} e^{-2\pi x^2} \, d\mu(x) = N_{2(n-l)} \leq a_n \, n! ,
\]

and that:

\[
e^{-2\pi} \int_{|x| \leq 1} d\mu \leq \int_{|x| \leq 1} e^{-2\pi x^2} \, d\mu(x) \leq N_0 < +\infty .
\]
This shows that conditions 5.5 and 5.6 are verified, ensuring that \( \mu \) is in fact a tempered measure.

Suppose now that there is a tempered measure \( \mu \) such that formula 5.10 is verified. Then, by theorem 15, conditions 5.4 and 5.5 are verified and so, there exists an \( l \in \mathbb{N} \) and a sequence \( (a_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^\mathbb{N} \) satisfying the summability condition given by \( \sum_{k=0}^{+\infty} (2\pi)^k a_k < +\infty \) such that:

\[
\forall k \in \mathbb{N} \quad \int_{|x| \geq 1} x^{2k-2l} e^{-2\pi x^2} \, d\mu(x) \leq a_k \, k!.
\]

With \( k = m + l \) and \( m \geq 0 \), this implies that:

\[
(5.11) \quad \forall m \in \mathbb{N} \quad \int_{|x| \geq 1} x^{2m} e^{-2\pi x^2} \, d\mu(x) \leq a_{m+l} \, (m + l)!
\]

For the remaining part of the integral, that is, the integral over the interval \( ]-1,1[ \) we have that for \( m \in \mathbb{N} \):

\[
(5.12) \quad \int_{|x| < 1} x^{2m} e^{-2\pi x^2} \, d\mu(x) \leq (m + l)! \int_{|x| < 1} \frac{x^{2m}}{(m + l)!} \, d\mu(x) = (m + l)! \, b_{m+l},
\]

where we have written \( b_{m+l} \) for the integral in the middle term of the chain of inequalities 5.12. As a straightforward application of Lebesgue monotone convergence theorem, we get:

\[
\sum_{m=0}^{+\infty} (2\pi)^{m+l} \, b_{m+l} \leq (2\pi)^l \sum_{m=0}^{+\infty} (2\pi)^m \int_{|x| < 1} \frac{x^{2m}}{m!} \, d\mu(x) = (2\pi)^l \int_{|x| < 1} \sum_{m=0}^{+\infty} \frac{(2\pi x^2)^m}{m!} \, d\mu(x) \leq (2\pi)^l \, e^{2\pi} \, \mu(\{|x| < 1\}) < +\infty.
\]

As a consequence, we have that: \( \sum_{m=0}^{+\infty} (2\pi)^m \, b_m < +\infty \). Finally, inequalities 5.11 and 5.12 imply that for \( m \in \mathbb{N} \):

\[
N_{2m} = \int_{\mathbb{R}} x^{2m} e^{-2\pi x^2} \, d\mu(x) \leq (a_{m+l} + b_{m+l}) \, (m + l)!.
\]

This inequality together with \( \sum_{m=0}^{+\infty} (2\pi)^m \, (a_m + b_m) < +\infty \), does prove proposition 1 in the theorem. \( \square \)

In what concerns the third component of our answer to problem 2, we are not aware of any result which could give some assurance that the solutions are always positive. In order to better understand the possible spectrum of behaviours of the solutions, we have performed some numerical essays using
the software Mathematica. The program used to perform the calculations is reproduced next.

\[
\begin{align*}
\tau &= 20 \\
\text{ad} &\{k, \text{Integer}\} := \text{ad}[k] = 1/k^7; \text{Table}[\text{ad}[k], \{k, 1, r\}] \\
g1 &\{n, \text{Integer}\} := g1[n] = (1/2) \cdot (1 - (-1)^{(n + 1)}) \cdot 2^{(1/2)} \cdot \text{Apply}[\text{Times}, \text{Table}[(2 \cdot k + 1)^2, \{k, 0, \text{Floor}[n/2 - 1]\}]] \\
a &\{m, n, \text{Integer}\} := a[m, k] = \text{If}[k \leq m, (1/((k!) \cdot (m - k)!)) \cdot g1[m - k, 0] \\
\text{Mtp}[l, \text{Integer}] := \text{Mtp}[l] = \text{Table}[a[m, k], \{m, 0, l\}, \{k, 0, l\}] \\
\text{Mtp}[r - 1] \\
\text{tbp}[p, \text{Integer}] := \text{tbp}[p] = \text{Simplify}[\text{LinearSolve}[	ext{Mtp}[p], \text{Table}[	ext{ad}[k], \{k, 1, p + 1\}]]] \\
\text{tbp}[r - 1] \\
N[\text{tbp}[r - 1]] \\
F[0, t] := \text{Exp}[-2 \cdot \pi \cdot t^2] \cdot ((1 - t)/(1 + t)) \\
F[n, t] := F[n, t] = \text{Simplify}[D[F[n - 1, t], t]] \\
\text{zer} &\{k, \text{Integer}\} := \text{Table}[0, \{i, 1, k\}] \\
\text{For}[i = 0, i < r, i++, F[i, t], PT1[t_] = \text{Simplify}[F[i, t]]; \\
\text{eq}[i] = PT1[0] \cdot \text{Exp}[2 \cdot \pi \cdot t^2]; \\
\text{Lit}[i] = \text{CoefficientList}[\text{Expand}[\text{eq}[i]], t^2]; \\
\text{Print}[\text{Lit}[i]] \\
\text{tad} = \text{Table}[[\text{Join}[	ext{Lit}[k], \text{zer}[r - k - 1]], \{k, 0, r - 1\}] \\
\text{pm}[1] := \text{tbp}[r - 1][1][1]/\text{tad}[1, 1] \\
\text{pm}[n, \text{Integer}] := pm[n] = (1/\text{tad}[[n, n]]) \cdot (\text{tbp}[r - 1][n]) - \\
\text{Apply}[\text{Plus}, \text{Table}[[\text{tad}[[n, k]] \cdot \text{pm}[k], \{k, 1, n - 1\}]]] \\
\text{Taf} = \text{Table}[[\text{Simplify}[	ext{pm}[n]], \{n, 1, r\}]] \\
N[\text{Taf}] \\
\text{Table}[[\text{N}[(2 \cdot \pi \cdot k)^k \cdot \text{pm}[k]/k!], \{k, 1, r\}] \\
\end{align*}
\]

Using this program, we have calculated for each of the following sequences defined for \( k \in \mathbb{N} \) by:

\[
\begin{align*}
M[k] &= 1, \quad M[k] = 1 + |\cos(k)|, \quad M[k] = k \\
M[k] &= \log(k)/k, \quad M[k] = 1/\sqrt{k}, \quad M[k] = k^2,
\end{align*}
\]

the solutions \( \text{pm}[k] \) for the linear systems 5.2 and 5.3 for \( N = 20 \). Next we considered the corresponding sequences given, for \( k \in \{1, \ldots, 20\} \) by:

\[
R[k] = \frac{\text{pm}[k] \cdot (2\pi)^k}{k!}.
\]

The sequences \( (\frac{R[k+1]}{R[k]})_{k \in \{1, \ldots, 19\}} \) define, in a natural way, piecewise linear functions. The corresponding plots are shown in figure 1. For a sequence such
as $M[k] = 1/k^2$, the solutions of the linear systems contained some negative terms and, as a consequence, can not be taken to represent even moments

![Graphs showing different functions of $M[k]$](image)

**Figure 1**

of a positive measure. A possible comment on the results just presented is that the necessary and sufficient condition given by theorem 17 is probably not verified for $M[k] = k^2$. So, for this sequence and for the sequence $M[k] = 1/k^2$ it is probably not possible to have corresponding tempered measures such that these sequences are the sequences of variances of the gaussian variables defined when the brownian distributions are applied to the Hermite functions. For the other sequences considered, theorem 17 is probably verified and so, the corresponding measures do probably exist.
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